ELEFTER 2022 Problem Set

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Good luck!

Contents

I Astrophysics

I.1 AGN (15 points)

It is believed that the accretion disk around supermassive black holes (BH) at galactic centres gives rise to UV thermal emission. This emission is associated with Active Galactic Nuclei (AGNs).

The optical spectra of bright AGNs show additional bright broad emission lines. Those emission lines arise from the dense gas in the Broad Line Region (BLR), which is ionized by the UV photons from the accretion disk. See Figure 1 to visualise this model.

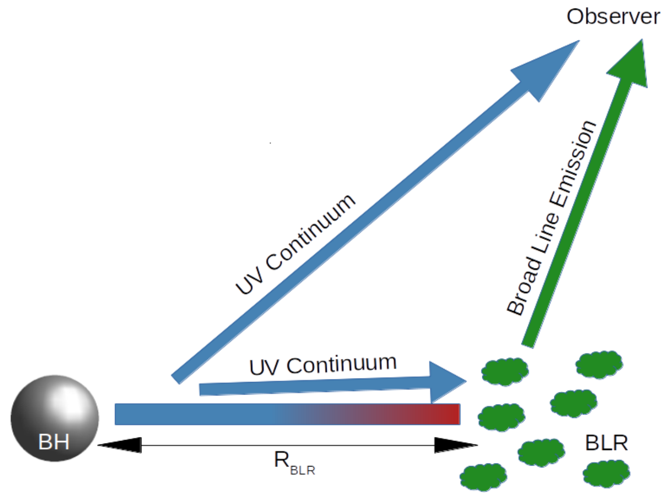


Figure 1

We can assume that the flux of broad emission lines varies in response to the variation of the UV continuum with a time delay. This time delay should be proportional to the separation R_{BLR} between the BH and the BLR.

Assume that the size of the accretion disk is negligible as compared to R_{BLR}.

I.1.1 [1 point(s)]

Estimate the time lag (days) between the B-band continuum and broad emission line H_β using the light curves shown in Figure 2. The x-axis is in reduced Julian Dates (JD).

I.1.2 [3 point(s)]

Estimate R_{BLR} in parsecs (pc).

I.1.3 [2 point(s)]

Estimate the angular separation of this region □BLR (in arcsec) from the blackhole, if this AGN is 100Mpc away from us. It is possible to estimate the mass of the system using the Virial theorem, if the velocity dispersion of the gasses in the BLR and the size of the system are known. Assume that the masses of the accretion disk and broad line region are negligible, as compared to the black hole. The velocity dispersion v_σ may

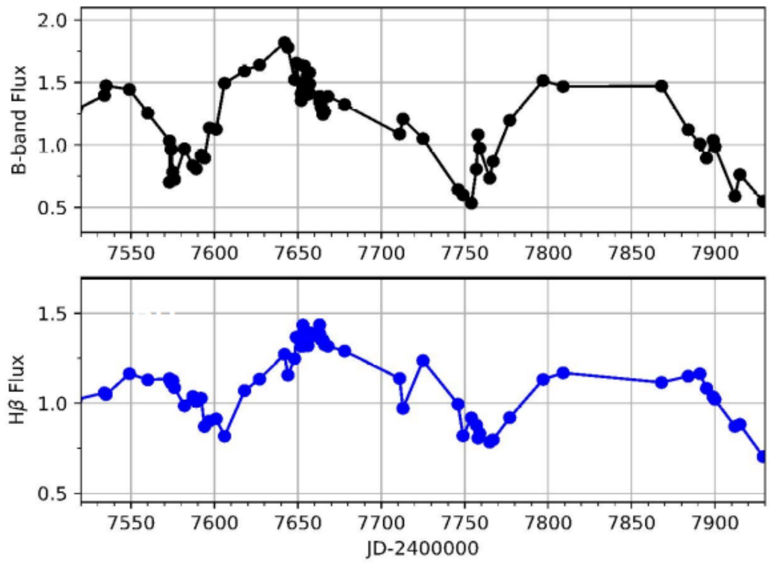


Figure 2

be estimated from the broadening of the given emission line. We will take the corresponding wavelength dispersion to be

$$\sigma = \frac{FWHM}{2.35}$$

where FWHM is the full width at half maximum of the broad emission line.

I.1.4 [5 point(s)]

Calculate the velocity dispersion v_σ in units of km s^{-1}, from the spectral line shown in Figure 3.

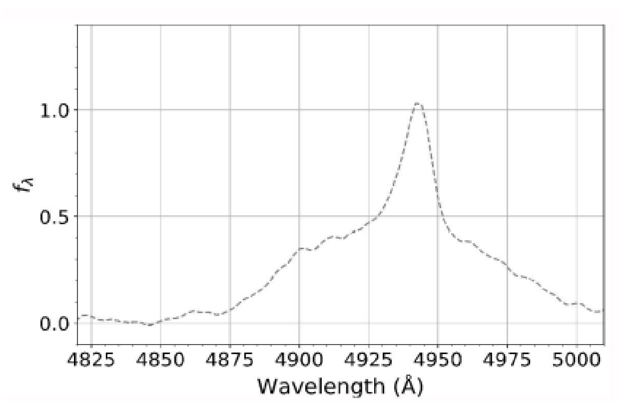


Figure 3

I.1.5 [4 point(s)]

Calculate the mass of the central BH ($M_{vir;BH}$) in a unit of $M_\odot$.

I.2 Mirror (10 points)

A bored cosmologist comes up with a thought experiment to determine the Hubble constant (H_0) for his model of a Steady-State-Universe. In this experiment, a large, fully reflecting flat mirror – carrying several gyroscopes that would maintain its spatial orientation in the same plane – would be placed at a distance D from the Solar System in a region without gravitational influences. From the Earth, a laser beam would be directed towards that region for a long period of time. After a time T, the radiation would return and be detected, allowing the determination of the fixed constant H_0.

I.2.1 [7 point(s)]

Find an expression for H_0 as a function of D, c (speed of light) and T. Consider that the separation S between the Solar System and the mirror increases only due to the expansion of the universe according to the law $S = se^{H_0 t}$, where s is the initial separation. You may use $e^x \approx 1 + x$ for $x \ll 1$, if necessary.

I.2.2 [3 point(s)]

Imagine that such a mirror is located in the vicinity of the star Vega. Vega was the first star outside the Solar System to be photographed and one of the first stars whose parallax (p = 0.125”) was accurately measured in 1840 by G. W. von Struve. Estimate the total duration of this H_0 measurement experiment.

I.3 Flat Earth (5 points)

A new model of the world is gaining in popularity among some people. These people believe in the "Flat Earth" view of the world, where the Earth is not a spheroid, but rather a circle with radius $R_{\oplus}$. The central axis of the Earth (normal to the circle passing through its centre C) is passes through the observer's zenith. This model must at least remain consistent with the observed phenomena, as listed below:

- The value of the solar constant is $S_{\odot} = 1366 W/m^2$
- The Earth's central axis precesses in a circle with a period 25800 years.
- The radius of the precession circle is 23.5°

We assume that the Earth is a perfect blackbody radiator and the Sun is sufficiently far away that all sun rays are parallel. Let us also assume that the Sun's current (initial) location is at the zenith.

I.3.1 [5 point(s)]

Determine how many years it will take for the Earth's equilibrium temperature to decrease by $1°°C$.

II Condensed Matter Physics

II.1 Two-site Problem (5 points)

Consider a potential representing two inequivalent wells separated by a barrier. In the limit of infinitely high barrier the two localized states have energies ε_1 and ε_2. For a finite barrier a fermion can tunnel between the states 1 and 2. Let the corresponding amplitude be τ. One can write down the Hamiltonian as

$$H = \varepsilon_1 c_1^\dagger c_1 + \varepsilon_2 c_2^\dagger c_2 - \tau(c_1^\dagger c_2 + c_2^\dagger c_1).$$

II.1.1 [2 point(s)]

Diagonalize the Hamiltonian and find its spectrum.

II.1.2 [3 point(s)]

Imagine that at $t = 0$ the fermion was localized in the state 1. Calculate the probability to find the electron in the same state at the moment t.

Hint

Consider the transition amplitude $\langle 0|c_1(t)c_1^\dagger(0)|0\rangle$, where $c_1(t) = e^{iHt}c_1 e^{-iHt}$.

II.2 Quantum Ising Model and Majorana Fermions (13 points)

Quantum Ising model (Also known as 1D Ising model in a transverse magnetic field) is a toy model famous for exhibiting quantum phase transition, i.e. transition not driven by temperature but instead driven by external magnetic field. We consider $T = 0$ case. The Hamiltonian of the model is:

$$\hat{\mathrm{H}}[\{\sigma\}] = J\left(-\lambda\sum_{n=1}^{N}\hat{\sigma}_n^x - \sum_{n=1}^{N-1}\hat{\sigma}_n^z\hat{\sigma}_{n+1}^z\right)^{1}. \tag{II.1}$$

$J > 0$ is the coupling constant of $z - z$ interaction and $\lambda \geq 0$ is a dimensionless parameter, that characterizes the amplitude of a transverse magnetic field $h = J\lambda$. $\hat{\sigma}_n^x$ and $\hat{\sigma}_n^z$ are the Pauli matrices for spin-$\frac{1}{2}$, each defined on a site physical site n of the chain. These Pauli matrices obey the known $SU(2)$ algebra relations when defined on the same site and commute when the site indices are different. Traditionally, we take $\hat{\sigma}_n^z$ to be a diagonal matrix, with eigenvalues of $+1$ (spin up, $|\uparrow\rangle_z$) and -1 (spin down, $|\downarrow\rangle_z$). We can also express spin up/down in x direction in the z basis as: $|\uparrow\rangle_x = |\uparrow\rangle_z + |\downarrow\rangle_z$ and $|\downarrow\rangle_x = |\uparrow\rangle_z - |\downarrow\rangle_z$. As an example, the states of this system expressed in the z basis can be written as

$$|\uparrow_1\rangle_z \otimes |\downarrow_2\rangle_z \otimes ... \otimes |\uparrow_j\rangle_z \otimes ... = |\uparrow_1, \downarrow_2, ..., \uparrow_j, ...\rangle_z \tag{II.2}$$

meaning that the first spin looks along z direction, the second in the opposite and etc. Similarly, for the x basis we have

$$|\uparrow_1\rangle_x \otimes |\downarrow_2\rangle_x \otimes ... \otimes |\uparrow_j\rangle_x \otimes ... = |\uparrow_1, \downarrow_2, ..., \uparrow_j, ...\rangle_x \tag{II.3}$$

II.2.1 [1 point(s)]

What is the global discrete symmetry of this problem? Express the unitary operator $\hat{U}$ in terms of Pauli matrices, under which

$$\hat{\sigma}_n^\alpha \to \hat{U}\hat{\sigma}_n^\alpha\hat{U}^\dagger, \qquad \text{so that} \qquad \hat{U}\hat{H}[\{\sigma\}]\hat{U}^\dagger = \hat{H}[\{\sigma\}].$$

II.2.2 [2 point(s)]

Argue what could be the ground state configurations of spins for $\lambda = 0$ and $\lambda = +\infty$, expressed as Eq.(II.2). What would happen if we used Eq.(II.3) instead? What are the possible values of the total spin along z direction (i.e. magnetization, interpreted as the so-called order parameter of the problem) in these limits and why should we expect a phase transition point at an intermediate value $\lambda = \lambda_c$?

Hint

The model possesses a global discrete symmetry.

II.2.3 [3 point(s)]

To extract a precise value of λ_c, we exploit a concept of self-duality. For convenience, let us denote links that connect sites n and $n+1$ as $n + \frac{1}{2}$ and define two new matrices in the following way

$$\hat{\mu}_{n+1/2}^z = \prod_{j=1}^{n}\hat{\sigma}_j^x, \qquad \hat{\mu}_{n+1/2}^x = \hat{\sigma}_n^z\hat{\sigma}_{n+1}^z. \tag{II.4}$$

[1] $\hat{\sigma}_n^z\hat{\sigma}_{n+1}^z \equiv \hat{\sigma}_n^z \otimes \hat{\sigma}_{n+1}^z$

$\hat{\mu}$ matrices satisfy exactly the same algebra as $\hat{\sigma}$. Eq.(II.4) is known as the duality transformation. The action of $\hat{\mu}^z_{n+1/2}$ on spin configuration is, for instance

$$\hat{\mu}^z_{n+1/2}|\uparrow_1, \uparrow_2, ..., \uparrow_{n-1}, \uparrow_n, ..., \uparrow_N\rangle_z = |\downarrow_1, \downarrow_2, ..., \downarrow_n, \uparrow_{n+1} ..., \uparrow_N\rangle_z, \tag{II.5}$$

meaning that $\hat{\mu}^z_{n+1/2}$ operator creates a domain wall in the spin configuration and thus disordering the system, hence the name - disorder operator. Invert the transformation Eq.(II.4), by expressing $\hat{\sigma}$ operators in terms of $\hat{\mu}$ operators and rewrite Eq.(II.1) Hamiltonian in terms of $\hat{\mu}$ operators, denoting it as $\hat{\mathrm{H}}[\{\mu\}]$. Let us forget for a second that initially we were working in the spin-up/down basis of $\hat{\sigma}^z$ matrices and assume that $\hat{\mathrm{H}}[\{\mu\}]$ is the starting Hamiltonian, therefore we work in spin-up/down basis of $\hat{\mu}^z$ instead. What are the possible values of the total disorder parameter in $\lambda \to +\infty$ and $\lambda = 0$ limits of $\hat{\mathrm{H}}[\{\mu\}]$ Hamiltonian? How does it compare with the same regimes for $\hat{\mathrm{H}}[\{\sigma\}]$? Extract the value of λ when the Hamiltonian maps to itself under the duality transformation, corresponding to the critical value λ_c. The Lee-Yang theorem justifies that there is only a single critical point in the model. Draw a phase diagram of $\hat{\mathrm{H}}[\{\sigma\}]$ quantum Ising model, indicating the region of order and disorder.

II.2.4 [3 point(s)]

The Jordan-Wigner transformation maps bosonic spin-$1/2$ operators onto the fermionic creation and annihilation operators in a very non-local fashion. Under the Jordan-Wigner transformation Eq.(II.1) exactly maps onto the model of a 1-dimensional p-wave superconductor (1DPS)

$$\hat{\mathrm{H}}[\{\sigma\}] \to \hat{\mathrm{H}}_{\mathrm{1DPS}} = -\lambda J \sum_{n=1}^{N}(2\hat{a}^\dagger_n \hat{a}_n - 1) - J \sum_{n=1}^{N-1}\left(\hat{a}^\dagger_n \hat{a}_{n+1} + \hat{a}^\dagger_{n+1}\hat{a}_n + \hat{a}^\dagger_n \hat{a}^\dagger_{n+1} + \hat{a}_{n+1}\hat{a}_n\right), \tag{II.6}$$

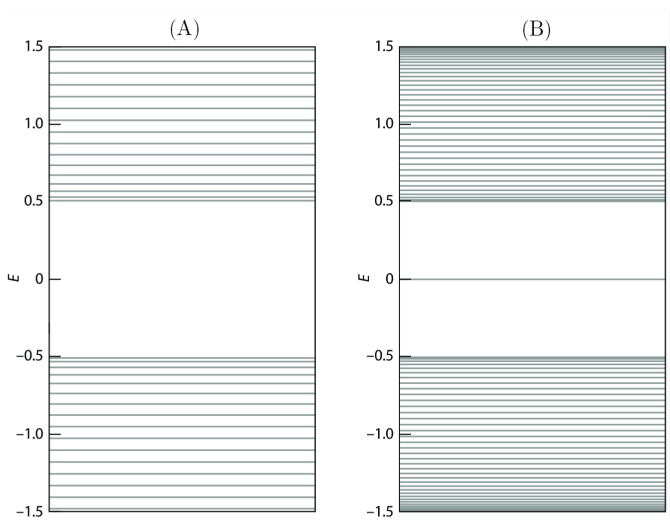
where $\hat{a}^\dagger_n$ and $\hat{a}_n$ are the creation and annihilation operators of spinless fermions at site n. These operators obey standard anti-commutation relations

$$\{\hat{a}^\dagger_n, \hat{a}_m\} = \delta_{nm}, \qquad \{\hat{a}_n, \hat{a}_m\} = 0. \tag{II.7}$$

The last two terms correspond to the superconducting coupling, creating and destroying two particles at a time. Due to this, the total particle number operator $\hat{Q} = \sum_{n=1}^{N} \hat{a}^\dagger_n \hat{a}_n$ does not commute with $\hat{\mathrm{H}}_{\mathrm{1DPS}}$ and the total particle number conservation is violated. However, the parity of the particle number is conserved - we either have odd or even number of particles in the system. The corresponding parity operator is

$$\hat{P} = e^{-i\pi\hat{Q}}. \tag{II.8}$$

$\hat{\mathrm{H}}_{\mathrm{1DPS}}$ model has a notorious feature in it's single-particle energy spectrum:



For $\lambda > \lambda_c$ (Same λ_c as it was in the Quantum Ising model) the spectrum of the $\hat{\mathrm{H}}_{\mathrm{1DPS}}$ is depicted on (A). However, as soon as $\lambda < \lambda_c$ condition is satisfied, depicted on (B), a mysterious energy level emerges - with energy exactly equal to zero! This corresponds to the so-called Majorana edge zero mode - a topologically protected mode, located at the left and right edges of the system. To get the essence of Majorana fermion, a simple analogy comes in handy: A complex number z can be split up as its real a and imaginary b parts, yielding $z = a + ib$. In a similar way, a creation and annihilation operator of a fermion can be represented as its "real" and "complex" parts as

$$\hat{a}_j = \frac{1}{2}\left(\hat{\zeta}_j - i\hat{\eta}_j\right), \qquad \hat{a}_j^\dagger = \frac{1}{2}\left(\hat{\zeta}_j + i\hat{\eta}_j\right), \tag{II.9}$$

where $\hat{\zeta}_j$ and $\hat{\eta}_j$ are the two Majorana fields. Derive the anti-commutation relations that the Majorana fields obey. Rewrite the Hamiltonian Eq.(II.6) and the Parity operator Eq.(II.8) in terms of the Majorana fields.

II.2.5 [4 point(s)]

Generally, Majorana zero mode $\hat{\Psi}$ is an operator with the following properties:

$$[\hat{H}, \hat{\Psi}] = 0, \quad \{\hat{P}, \hat{\Psi}\} = 0, \quad \hat{\Psi}^\dagger\hat{\Psi}|_{N\to\infty} = 1. \tag{II.10}$$

For a zero mode to be also an edge mode, it must be localized at the boundaries of the system. Let us denote such right and left edge zero mode operator as $\hat{\Psi}_R$ and $\hat{\Psi}_L$. The matrix elements of $\hat{\Psi}_{R,L}$ must decay exponentially as we move l distance away from the corresponding boundary. If $\lambda = 0$, then $\hat{\zeta}_1$ and $\hat{\eta}_N$ do not appear in the Hamiltonian at all, they are completely isolated from the rest of the system. They also anti-commute with the parity operator and satisfy the normalization condition. Since both of them are localized at the left and right edges of the chain, they are an exact edge zero-mode operators

$$\hat{\Psi}_L(\lambda = 0) = \hat{\zeta}_1, \qquad \hat{\Psi}_R(\lambda = 0) = \hat{\eta}_N. \tag{II.11}$$

The 1DPS is in a topologically non-trivial state when $|\lambda| < \lambda_c$, therefore we need to check if the edge zero-mode operators persist as we deviate from $\lambda = 0$ point and see how does it get modified. For concreteness, let us concentrate on the left edge mode operator $\hat{\Psi}_L$ only. Develop an <u>iterative</u> method to write down the expression for $\hat{\Psi}_L^n(\lambda)$, n indicating that Eq.(II.11) has been corrected upto n order in λ. After the last possible step of the iteration, under what condition can we take $[\hat{H}_{\mathrm{1DPS}}, \hat{\Psi}_L^{N-1}] = 0$? What about $\hat{\Psi}_R$? What happens to both modes when $|\lambda| > \lambda_c$?

Hint

Iteration method - Suppose we have two operators $\hat{A} = \hat{B} + \hat{C}$ and $\hat{Z}$, such that $[\hat{B}, \hat{Z}] = 0$, but $[\hat{C}, \hat{Z}] = \hat{D}$. Find $\hat{Z}'$, such that $[\hat{B}, \hat{Z}'] = -\hat{D}$ and add it to the initial $\hat{Z}$. Repeat.

II.3 AKLT Model (12 points)

We consider the following Hamiltonian

$$H_{\text{AKLT}} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + K \sum_i \left(\vec{S}_i \cdot \vec{S}_{i+1} \right)^2 ,$$

where $\vec{S}_i = (S_i^x, S_i^y, S_i^z)$ is the vector spin-1 operator at site i.
Each site has local Hilbert space with three states: $|-1\rangle, |0\rangle, |+1\rangle$.

II.3.1 [2 point(s)]

Consider an operator $\hat{A}$ with discrete eigenvalues a_n. Define $\hat{P}^{(m)}$ as

$$\hat{P}^{(m)} = C \prod_{n \neq m} \left(\hat{A} - a_n \right) ,$$

where C is a normalization constant. Show that $\hat{P}^{(m)}$ acts as a projection operator and determine the normalization constant C.

Hint

Projector satisfies

$$\hat{P}^{(m)} |\psi_n\rangle = \begin{cases} 0 & \text{for } n \neq m, \\ |\psi_n\rangle & \text{for } n = m. \end{cases}$$

II.3.2 [2 point(s)]

Consider two neighbouring sites i and $i+1$. What are the possible eigenvalues of the total spin operator $\vec{S}_{\text{tot}}^2 = \left(\vec{S}_i + \vec{S}_{i+1} \right)^2$?

Hint

How does one combine two spin-1 degrees of freedom?

II.3.3 [2 point(s)]

Using the results of the first two sub-problems construct a projection operator $P_{i,i+1}^{(2)}$ that projects onto the total spin-2 subspace of the combined spin-1 degrees of freedom at sites i and $i+1$. Express it in terms of $\vec{S}_i$ and $\vec{S}_{i+1}$. What is the relationship between this projector and the AKLT Hamiltonian? What is the ground state energy of the AKLT model?

Hint

$(\vec{S})^2 = 2$ for spin 1.

II.3.4 [2 point(s)]

Consider two sites i and $i+1$ and split the spin-1 degrees of freedom into two spin-1/2 degrees of freedom. The two sites combined will now have 4 spin-1/2 degrees of freedom. How can we combine these 4 degrees of freedom in order to minimize the AKLT Hamiltonian for this pair of sites? Write down the associated ground state wavefunction $|\Psi_0\rangle_{i,i+1}$ using spin-1/2 states, $|\alpha,\beta\rangle_i |\gamma,\delta\rangle_{i+1}$, where $\alpha,\beta,\gamma,\delta$ take the values $\uparrow,\downarrow$.

Hint

For instance, state $|\uparrow,\uparrow\rangle_i |\uparrow,\uparrow\rangle_{i+1}$ would mean all four spin-1/2 projections are spin-up, i.e. $S^z = +1/2$.

II.3.5 [2 point(s)]

Construct an operator $\hat{T}_i$ that converts from the spin-1/2 triplet basis to the spin-1 basis, e.g. $\hat{T}_i |\uparrow,\uparrow\rangle_i = |+1\rangle_i$.

Hint

One can write such operator in the form $\hat{T}_i = t^{\sigma}_{\alpha,\beta} |\sigma\rangle_i \langle\alpha,\beta|_i$, where $\alpha,\beta =\uparrow,\downarrow$ and $\sigma = -1, 0, 1$.

II.3.6 [2 point(s)]

Combining the results of II.3.4 and II.3.5 write down the ground state wavefunction of the AKLT model. State the difference between periodic and open boundary conditions. What is unusual about the edges in the case of open boundary conditions?

III Particle Physics

III.1 Born Approximation (10 points)

III.1.1 [1 point(s)]

For a particle with mass m, the first Born approximation is defined as

$$f^{(1)}(\mathbf{k}',\mathbf{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}V(\mathbf{x}),$$

where $V(\mathbf{x})$ is the scattering potential. Show, that for a spherically symmetric potential this simplifies to

$$f^{(1)}(\mathbf{k}',\mathbf{k}) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rdr\ \sin(qr)V(r).$$

The scattering is elastic.

III.1.2 [2 point(s)]

A particle of mass m is scattered in the Yukawa potential:

$$V(r) = \frac{V_0}{r}e^{-\kappa r}.$$

Using the result above calculate the differential cross-section in the first Born approximation.

III.1.3 [1 point(s)]

For what values of κ and V_0 is the Born approximation reasonable at low energies?

III.1.4 [1 point(s)]

In the limit $\kappa \to 0$ Yukawa potential transforms into Coulomb interaction. Show that the cross-section (or rather, the first Born approximation) describes Rutherford scattering in this limit.

III.1.5 [2 point(s)]

The second Born amplitude is defined as

$$f^{(2)}(\mathbf{k}',\mathbf{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\left\langle \mathbf{k}'\left| V\frac{1}{E-H_0+i\varepsilon}V\right|\mathbf{k}\right\rangle.$$

Show that the forward scattering amplitude for the Yukawa potential is given by

$$f^{(2)}(\mathbf{k},\mathbf{k}) = -4\pi\left(\frac{2m}{\hbar^2}\right)^2\frac{V_0^2}{(2\pi)^3}4\pi\int_0^\infty\frac{\tilde{k}^2d\tilde{k}}{(k^2-\tilde{k}^2+i\epsilon)(\kappa^2+(k-\tilde{k})^2)(\kappa^2+(k+\tilde{k})^2)}$$

III.1.6 [2 point(s)]

Identify all the poles of the integrand in the above result and integrate it over all $\tilde{k}$ to obtain

$$f^{(2)}(\mathbf{k}, \mathbf{k}) = \left(\frac{2m}{\hbar^2}\right)^2 \frac{V_0^2}{2\kappa^2(\kappa - 2ik)}.$$

III.1.7 [1 point(s)]

The optical theorem relates the full cross-section to the imaginary part of the forward scattering amplitude. State the optical theorem and check that it holds for the Yukawa potential (the first terms in powers of V_0). Why is the second Born approximation needed for this?

III.2 The Higgs Mechanism (10 points)

Consider the following Lagrangian:

$$\mathcal{L} = (D^\mu \Phi)^\dagger D_\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda(\Phi^\dagger \Phi)^2$$

where

- $\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$ is an $SU(2)$ doublet;
- $D_\mu = \partial_\mu + ig\frac{\tau^a}{2} W_\mu^a$ is the covariant derivative;
- τ^a denote the Pauli matrices (see the Appendix below), $a = 1, 2, 3$;
- W_μ^a are vector bosons;
- g is a coupling constant.

III.2.1 [1 point(s)]

Under the local $SU(2)$ transformations

$$\Phi \to e^{i\alpha^a(x)\frac{\tau^a}{2}} \Phi,$$

the vector fields transform as

$$W_\mu^a \to W_\mu^a - \frac{1}{g} \partial_\mu \alpha^a(x) - \epsilon^{abc} \alpha^b(x) W_\mu^c.$$

ϵ^{abc} is the totally-antisymmetric symbol with $\epsilon^{123} = 1$.
Show that a mass term for the vector bosons breaks the gauge invariance of the Lagrangian.

III.2.2 [1 point(s)]

We assume $\lambda > 0$, so that the potential

$$V = \mu^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2$$

is bounded from below.

Which case describes a theory with spontaneous symmetry breaking: $\mu^2 > 0$ or $\mu^2 < 0$?

III.2.3 [1 point(s)]

What conditions must the fields $\Phi, \Phi^\dagger$ satisfy in order to minimize V?

III.2.4 [1 point(s)]

For the ground state we choose

$$\Phi_0 = \begin{pmatrix} 0 \\ v \end{pmatrix}.$$

In other words, we set $\phi_1 = \phi_2 = \phi_3 = 0$ and $\phi_4 = v = \text{const}$. Why are we allowed to do this? What is the value for the constant v?

III.2.5 [1 point(s)]

We expand the fields around Φ_0:

$$\Phi = \Phi_0 + \Delta\Phi = \begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} \Delta\phi_1(x) + i\Delta\phi_2(x) \\ \Delta\phi_3(x) + i\Delta\phi_4(x) \end{pmatrix} = \begin{pmatrix} \Delta\phi_1(x) + i\Delta\phi_2(x) \\ v + \Delta\phi_3(x) + i\Delta\phi_4(x) \end{pmatrix}$$

Show that this is equivalent to the infinitesimal transformation

$$\Phi = \frac{1}{\sqrt{2}} e^{i\frac{\theta^a(x)}{v}\tau^a} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}.$$

How are the fields $\Delta\phi_1, \Delta\phi_2, \Delta\phi_3, \Delta\phi_4$ given in terms of $\theta_1, \theta_2, \theta_3, h$?

III.2.6 [1 point(s)]

Consider the kinetic part of the Lagrangian:

$$\mathcal{L}_{kin} = (D^\mu\Phi)^\dagger(D_\mu\Phi)$$

Show that inserting

$$\Phi = \frac{1}{\sqrt{2}} e^{i\frac{\theta^a(x)}{v}\tau^a} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}.$$

into $\mathcal{L}_{kin}$ gives

$$\begin{aligned}
\mathcal{L}_{kin} = {} & \frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{1}{2}(\partial^\mu\theta_1)(\partial_\mu\theta_1) + \frac{1}{2}(\partial^\mu\theta_2)(\partial_\mu\theta_2) + \frac{1}{2}(\partial^\mu\theta_3)(\partial_\mu\theta_3) \\
& + \frac{g}{2}W^1_\mu\left(h\partial^\mu\theta_1 + v\partial^\mu\theta_1 - \theta_1\partial^\mu h + \theta_3\partial^\mu\theta_2 - \theta_2\partial^\mu\theta_3\right) \\
& + \frac{g}{2}W^2_\mu\left(h\partial^\mu\theta_2 + v\partial^\mu\theta_2 - \theta_2\partial^\mu h + \theta_1\partial^\mu\theta_3 - \theta_3\partial^\mu\theta_1\right) \\
& + \frac{g}{2}W^3_\mu\left(h\partial^\mu\theta_3 + v\partial^\mu\theta_3 - \theta_3\partial^\mu h + \theta_2\partial^\mu\theta_1 - \theta_1\partial^\mu\theta_2\right) \\
& + \frac{g^2}{8}\left(\left(W^1_\mu\right)^2 + \left(W^2_\mu\right)^2 + \left(W^3_\mu\right)^2\right)\left(v^2 + 2vh + h^2 + \theta_1^2 + \theta_2^2 + \theta_3^2\right)
\end{aligned}$$

III.2.7 [1 point(s)]

Consider the potential part of the Lagrangian:

$$V = \mu^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2 = -\lambda v^2 \Phi^\dagger \Phi + \lambda(\Phi^\dagger \Phi)^2$$

Show that inserting

$$\Phi = \frac{1}{\sqrt{2}} e^{i \frac{\theta^a(x)}{v} \tau^a} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}.$$

into V gives

$$V = \frac{\lambda}{4} \left(h^4 + 4h^3 v - v^4 + 4hv(\theta_1^2 + \theta_2^2 + \theta_3^2) + (\theta_1^2 + \theta_2^2 + \theta_3^2)^2 + 4h^2 v^2 + 2h^2(\theta_1^2 + \theta_2^2 + \theta_3^2) \right).$$

III.2.8 [1 point(s)]

Examine the whole resulting Lagrangian. How does the number of degrees of freedom compare to that of the initial Lagrangian? What is the reason for this and how can it be resolved?

III.2.9 [1 point(s)]

Use gauge freedom to eliminate the θ fields completely from the Lagrangian.

III.2.10 [1 point(s)]

What are the masses of the vector bosons after the elimination of the θ fields? How many degrees of freedom does the resulting Lagrangian have?

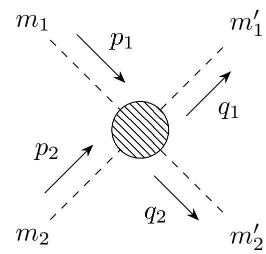
Appendix

Pauli matrices:

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

III.3 Electron-Positron to Pions (10 points)

Part 1: Kinematics



For a $2 \to 2$ process of spinless particles with initial momenta p_1, p_2 and final momenta q_1, q_2, the amplitude can depend only on the scalar products:

$$p_1^2,\ p_2^2,\ q_1^2,\ q_2^2,\ p_1 \cdot p_2,\ p_1 \cdot q_1,\ p_1 \cdot q_2,\ p_2 \cdot q_1,\ p_2 \cdot q_2,\ q_1 \cdot q_2.$$

III.3.1 [1 point(s)]

Give arguments why only 2 of these 10 scalars are independent. Where do the constraints come from?

III.3.2 [1 point(s)]

The n-particle phase space is defined as

$$d\Phi_n = \delta^{(4)}\left(\sum_i p_i - \sum_j q_j\right) \prod_{j=1}^{n} \frac{d^3 q_j}{(2\pi)^3 2E_{\vec{q}_j}},$$

The differential cross section for a $2 \to 2$ process is

$$d\sigma_{m_1 m_2 \to m_1' m_2'} = \frac{(2\pi)^4 \left|\langle q_1, q_2 | t | p_1, p_2 \rangle\right|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} d\Phi_2$$

Show that the full cross section is then

$$\sigma_{m_1 m_2 \to m_1' m_2'} = \frac{1}{64\pi^2} \frac{\sqrt{\lambda(s, m_1'^2, m_2'^2)}}{\sqrt{\lambda(s, m_1^2, m_2^2)}} \frac{1}{s} \int \frac{|t_{fi}|^2}{\mathbb{S}} d\Omega_{\vec{q}}$$

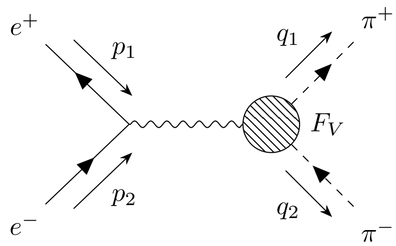
where $\mathbb{S}$ is the symmetry factor and λ is the Källén function, defined as

$$\lambda\left(s, m_1^2, m_2^2\right) = \left(s - (m_1 + m_2)^2\right)\left(s - (m_1 - m_2)^2\right).$$

t_{fi} is the invariant amplitude of the process (the indices i and f stand for initial and final states, respectfully).

Part 2: $e^+e^- \to \pi^+\pi^-$

We consider the process $e^+e^- \to \pi^+\pi^-$, described by the following diagram:



We define the Mandelstam variables as follows:

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2,$$
$$t = (p_1 - q_1)^2 = (p_2 - q_2)^2,$$
$$u = (p_1 - q_2)^2 = (p_2 - q_1)^2.$$

Apart from that, let us define

$$k = p_1 + p_2 = q_1 + q_2,$$
$$l = p_1 - p_2, \quad l' = q_1 - q_2.$$

III.3.3 [1 point(s)]

Give the expression for the leptonic current L^μ (left side of the diagram above) using the Feynman rules for QED.

III.3.4 [1 point(s)]

The hadronic current H^μ (right side of the diagram above) can be written as

$$H^\mu = (q_1 + q_2)^\mu G_V(s) + (q_1 - q_2)^\mu F_V(s).$$

Argue, why $G_V(s)$ can be safely neglected here.

III.3.5 [1 point(s)]

Give the expression for the invariant amplitude $\mathcal{M}$ for the process.

III.3.6 [1 point(s)]

Square the invariant amplitude, average out over all initial spins and sum over all final ones. Give the final result for the spin-averaged invariant matrix element squared $\overline{|\mathcal{M}|^2}$.

III.3.7 [1 point(s)]

Calculate the following trace

$$\frac{1}{4}\mathrm{tr}\left(\gamma^\mu \left(\not{p}_1 - m_e\right) \gamma^\nu \left(\not{p}_2 + m_e\right)\right)$$

Hint

$\not{A} = \gamma^\mu A_\mu$.

III.3.8 [1 point(s)]

Express $\overline{|\mathcal{M}|^2}$ in terms of the Mandelstam variable s, the scattering angle θ_s, and the Källén function λ, where

$$\begin{aligned}\cos(\theta_s) &= \frac{t-u}{\kappa(s)},\\ \kappa(s) &= \frac{\lambda^{1/2}(s, m_\pi^2, m_\pi^2)\lambda^{1/2}(s, m_e^2, m_e^2)}{s},\\ \lambda\left(s, m_1^2, m_2^2\right) &= \left(s - (m_1+m_2)^2\right)\left(s - (m_1-m_2)^2\right)\end{aligned}$$

III.3.9 [1 point(s)]

Integrate $\overline{|\mathcal{M}|^2}$ over the solid angle to obtain $\int \overline{|\mathcal{M}|^2} d\Omega$.

III.3.10 [1 point(s)]

Calculate the total cross section $\sigma_{e^+e^-\to\pi^+\pi^-}$.

Hint

You may use the limit $m_e^2 \ll s$.

IV Other

IV.1 Breaking Classical Mechanics (5 points)

Construction of Quantum mechanics from Classical mechanics usually begins with a process known as Quantization. This is usually done by constructing a map which takes observables to operators, that is:

$$\{f, g\} \to -\frac{i}{\hbar}[\hat{f}, \hat{g}]$$

Where $\{-,-\}$ is the Poisson bracket and $[-,-]$ is the commutator. One of the common properties of these brackets is that they form a Lie algebra, that is, they satisfy the following properties:

1. The bracket $[-,-]$ is billinear.
2. For any f, g we have $[f, g] = -[g, f]$
3. The bracket satisfies Jacobi identity, that is, for any f, g, h we have:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

IV.1.1 [5 point(s)]

Suppose that we have a "broken" classical mechanics in 3 dimensions, where $\{-,-\}$ doesn't satisfy the Jacobi identity. Prove that the resulting quantum mechanics would violate Heisenberg's uncertainty principle.

IV.2 The Green's Function (5 points)

Consider the Green's function in three dimensions:

$$G(\vec{x}) = -\frac{e^{ik|\vec{x}|}}{4\pi|\vec{x}|}.$$

IV.2.1 [2 point(s)]

Show $(\Delta + k^2)G(\vec{x}) = 0$ for $\vec{x} \neq 0$, where Δ is the Laplace operator.

IV.2.2 [3 point(s)]

Show that $G(\vec{x})$ satisfies the inhomogeneous differential equation

$$(\Delta + k^2)G(\vec{x}) = \delta^{(3)}(\vec{x}).$$

Hint

Consider the integral

$$\int_{|\vec{x}|\leq 1} d^3x(\Delta + k^2)G(\vec{x}).$$

IV.3 Point group D_6 (5 points)

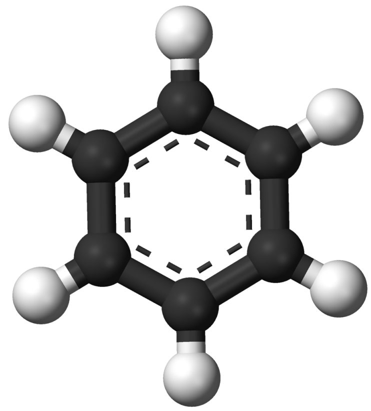


Figure 4: A molecule with D_6 symmetry. Credits: Wikipedia.

Consider the dihedral group $D_6 = \langle b, c \mid b^2 = c^6 = (bc)^2 = e \rangle$, which is the symmetry group for an unoriented hexagon.

IV.3.1 [1 point(s)]

D_6 has 6 conjugacy classes. One element per class is given below:

$$
\begin{aligned}
\mathcal{C}\ell_1 &= \{e, \dots\}\,, \\
\mathcal{C}\ell_2 &= \{c, \dots\}\,, \\
\mathcal{C}\ell_3 &= \left\{c^2, \dots\right\}, \\
\mathcal{C}\ell_4 &= \left\{c^3, \dots\right\}, \\
\mathcal{C}\ell_5 &= \{b, \dots\}\,, \\
\mathcal{C}\ell_6 &= \{bc, \dots\}\,.
\end{aligned}
$$

Complete the classes by adding corresponding elements within. Show why a specific element should belong to a specific class.

Hint

Not all $\{\dots\}$ are meant to be filled.

IV.3.2 [2 point(s)]

Let $\nu = 1, \dots, 6$ enumerate irreducible representations of D_6 and d_ν denote the dimension of the representation. We consider a 6-dimensional representation $D^{(7)}$. The characters for the irreducible representations are given in Table 1. Fill the table by calculating the characters for $D^{(7)}$.

Hint

Don't get confused by the notation: D_6 stands for the dihedral group. $D^{(\nu)}$ stand for specific representations.

$D^{(\nu)}$	d_ν	$\mathcal{C}\ell_1$	$\mathcal{C}\ell_2$	$\mathcal{C}\ell_3$	$\mathcal{C}\ell_4$	$\mathcal{C}\ell_5$	$\mathcal{C}\ell_6$
$D^{(1)}$	1	1	1	1	1	1	1
$D^{(2)}$	1	1	-1	1	-1	-1	1
$D^{(3)}$	1	1	-1	1	-1	1	-1
$D^{(4)}$	1	1	1	1	1	-1	-1
$D^{(5)}$	2	2	1	-1	-2	0	0
$D^{(6)}$	2	2	-1	-1	2	0	0
$D^{(7)}$	6	...	...	...	...	...	...

Table 1: Character table for D_6.

IV.3.3 [1 point(s)]

Using the characters derived in the previous section, decompose $D^{(7)}$ into irreducible representations.

IV.3.4 [1 point(s)]

Consider a molecule with D_6 symmetry, which transforms under $D^{(7)}$ (an example is given on Figure 4). What can you deduce about the energy levels (and their degeneracies) of this molecule, judging from the decomposition of $D^{(7)}$?

IV.4 Dirac Equation: Angular Momentum and Parity (5 points)

The Dirac equation for spin-1/2 particles is written as

$$i\partial_t \psi = \hat{H}\psi,$$

where

$$\hat{H} := \alpha^i \left(\hat{p}_i - qA_i\right) + \beta m + \mathbb{I} q\Phi.$$

Here $\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$.

We assume that the electric field is time-independent and rotationally invariant:

$$V(\mathbf{x}) := q\Phi = V(r).$$

We take the vector potential to be vanishing: $A_i = 0$. This simplifies the Hamiltonian to

$$\hat{H} = \alpha^i \hat{p}_i + \beta m + \mathbb{I} V$$

We combine the angular momentum and the spin operators

$$\hat{L}_i = \epsilon_{ijk} \hat{x}_j \hat{p}_k$$
$$S_i = \frac{1}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

to obtain the total angular momentum operator

$$\hat{J}_i = \mathbb{I}\hat{L}_i + S_i.$$

IV.4.1 [1 point(s)]

Show that the commutation relations for $\hat{J}$ are

$$\left[\hat{J}_i, \hat{J}_j\right] = i\epsilon_{ijk}\hat{J}_k, \qquad \left[\hat{J}_i, \hat{J}^2\right] = 0$$

IV.4.2 [2 point(s)]

Show that $\hat{J}_i$ and $\hat{J}^2$ commute with the given Hamiltonian:

$$\left[\hat{H}, \hat{J}_i\right] = 0, \qquad \left[\hat{H}, \hat{J}^2\right] = 0.$$

IV.4.3 [2 point(s)]

Parity operator for spinors is defined as $\hat{P}_s := \beta\hat{P}$ and acts as

$$\hat{P}_s \psi(t, \mathbf{x}) = \beta\psi(t, -\mathbf{x})$$

Show that this operator commutes with $\hat{H}$, $\hat{J}_i$, and $\hat{J}^2$.

IV.5 Transverse Magnetic Susceptibility of an Isotropic Ferromagnet (5 points)

In an isotropic ferromagnet, the ground state with all the spins polarized in the same direction is infinitely degenerate. The ground state manifold represents a sphere whose points correspond to possible directions of the spontaneous magnetization

$$\vec{M} = N^{-1} \sum_{i=1}^{N} \langle \vec{S}_i \rangle \, .$$

In an external magnetic field $\vec{h}_0$ the magnetization $\vec{M}$ will be aligned along $\vec{h}_0$. A small transverse magnetic field $\vec{h}_\perp$ (with $\vec{h}_\perp \cdot \vec{h}_0 = 0$) will slightly change the direction of $\vec{M}$.

IV.5.1 [3 point(s)]

Calculate the transverse magnetic susceptibility of the ferromagnet

$$\chi_\perp(h_0) = \lim_{h_\perp \to 0} \frac{\partial M(h_0; h_\perp)}{\partial h_\perp}.$$

IV.5.2 [2 point(s)]

What is the property of $\chi_\perp$ in the limit $h_0 = 0$? Explain the result.

IV.6 Gaussian Integrals (5 points)

IV.6.1 [2 point(s)]

Let A be a real, symmetric, positive definite matrix. Show the following identity for multi-dimensional integrals over real variables x_i:

$$\int \prod_{i=1}^{n} dx_i \,\mathrm{exp}\left(-\frac{1}{2}x_k A_{kl} x_l + J_k x_k\right) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}}\,\mathrm{exp}\left(\frac{1}{2}J_k A^{-1}_{jl} J_l\right).$$

IV.6.2 [3 point(s)]

Show that for complex variables z_i, the previous result can be generalized as follows:

$$\int \prod_{i=1}^{n} dz_i^* dz_i \,\mathrm{exp}\left(-z_k^* H_{kl} z_l + J_k^* z_k + J_k z_k^*\right) = \frac{(2\pi i)^n}{\det H}\,\mathrm{exp}\left(J_k^* H^{-1}_{kl} J_l\right)$$

where H is now hermitian, positive definite matrix.