ELEFTER 2022 Problem Set

G. Chanturia[1,2], G. Butbaia[3], T. Meshveliani[4], L. Razmadze[1,5], M. Tsitsishvili[6,7], and B. Beradze[8,9]

[1]University of Bonn, Bonn, Germany
[2]Helmholtz Institute for Radiation and Nuclear Physics, Bonn, Germany
[3]University of New Hampshire, Durham, United States
[4]University of Iceland, Reykjavik, Iceland
[5]Jülich Center for Hadron Physics, Forschungszentrum Jülich, Germany
[6]International Centre for Theoretical Physics, Trieste, Italy
[7]International School for Advanced Studies, Trieste, Italy
[8]Andronikashvili Institute of Physics, Tbilisi, Georgia
[9]Ilia State University, Tbilisi, Georgia

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Good luck!

Contents

I Astrophysics

I.1 AGN (15 points)

It is believed that the accretion disk around supermassive black holes (BH) at galactic centres gives rise to UV thermal emission. This emission is associated with Active Galactic Nuclei (AGNs).

The optical spectra of bright AGNs show additional bright broad emission lines. Those emission lines arise from the dense gas in the Broad Line Region (BLR), which is ionized by the UV photons from the accretion disk. See Figure 1 to visualise this model.

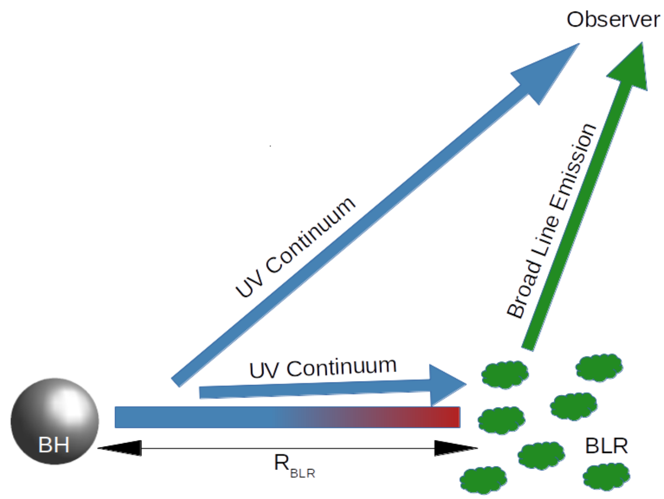


Figure 1

We can assume that the flux of broad emission lines varies in response to the variation of the UV continuum with a time delay. This time delay should be proportional to the separation R_{BLR} between the BH and the BLR.

Assume that the size of the accretion disk is negligible as compared to R_{BLR}.

I.1.1 [1 point(s)]

Estimate the time lag (days) between the B-band continuum and broad emission line H_β using the light curves shown in Figure 2. The x-axis is in reduced Julian Dates (JD).

Solution

Taking multiple reference points, we get that the BLR emission lags by about 20-25 days. Answers from 15 to 25 days are acceptable.

I.1.2 [3 point(s)]

Estimate R_{BLR} in parsecs (pc).

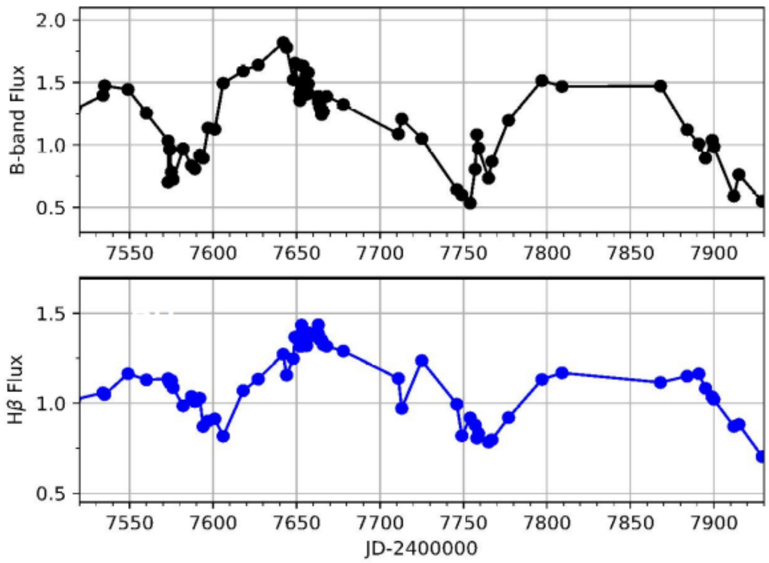


Figure 2

Solution

As the AGN is located very far, the time lag can be approximated as the time taken by UV emission to reach BLR region. Thus,

$$R_{BLR} = c\Delta t = 3 \times 10^8 \times 20 \times 86400 = 5.2 \times 10^{14}\text{m} = (0.017 \pm 0.004)\text{pc}.$$

I.1.3 [2 point(s)]

Estimate the angular separation of this region BLR (in arcsec) from the blackhole, if this AGN is 100Mpc away from us. It is possible to estimate the mass of the system using the Virial theorem, if the velocity dispersion of the gasses in the BLR and the size of the system are known. Assume that the masses of the accretion disk and broad line region are negligible, as compared to the black hole. The velocity dispersion v_σ may be estimated from the broadening of the given emission line. We will take the corresponding wavelength dispersion to be

$$\sigma = \frac{FWHM}{2.35}$$

where FWHM is the full width at half maximum of the broad emission line.

Solution

As the AGN is 100 MPC away from us,

$$\theta_{BLR} = \frac{0.017}{100 \times 10^6} \times 206265 = (3.5 \pm 0.9) \times 10^{-5}\text{arcsec}$$

I.1.4 [5 point(s)]

Calculate the velocity dispersion v_σ in units of km s^{-1}, from the spectral line shown in Figure 3.

Solution

The FWHM is approximately $(85 \pm 5)A$ and the peak is approximately at $(4940 \pm 5)A$

$$v_\sigma = \frac{\sigma c}{\lambda_{peak}} = \frac{FWHM \times c}{2.35\lambda_{peak}} = \frac{85 \times 3 \times 10^5}{2.35 \times 4940}$$

$$v_\sigma = (2200 \pm 140) kms^{-1}$$

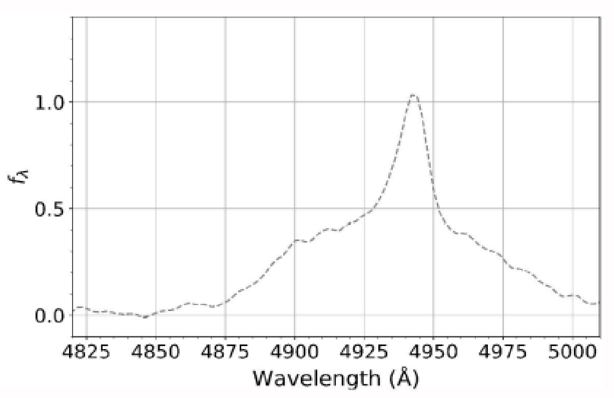


Figure 3

I.1.5 [4 point(s)]

Calculate the mass of the central BH ($M_{vir;BH}$) in a unit of $M_\odot$.

Solution

$$M_{vir,BH} = \frac{v_\sigma^2 R_{BLR}}{G} = \frac{(2.2 \times 10^6)^2 \times 5.2 \times 10^14}{6.674 \times 10^{-11}} = 3.8 \times 10^{37} kg$$
$$M_{vir,BH} = (1.9 \pm 0.7) \times 10^7 M_\odot$$

I.2 Mirror (10 points)

A bored cosmologist comes up with a thought experiment to determine the Hubble constant (H_0) for his model of a Steady-State-Universe. In this experiment, a large, fully reflecting flat mirror – carrying several gyroscopes that would maintain its spatial orientation in the same plane – would be placed at a distance D from the Solar System in a region without gravitational influences. From the Earth, a laser beam would be directed towards that region for a long period of time. After a time T, the radiation would return and be detected, allowing the determination of the fixed constant H_0.

I.2.1 [7 point(s)]

Find an expression for H_0 as a function of D, c (speed of light) and T. Consider that the separation S between the Solar System and the mirror increases only due to the expansion of the universe according to the law $S = se^{H_0 t}$, where s is the initial separation. You may use $e^x \approx 1 + x$ for $x \ll 1$, if necessary.

Solution

Let t_1 be the time taken by the light beam from the Solar System to the mirror, let t_2 be the time taken by the beam from the mirror to the Solar System and T the total time to go back and forth. As a first order approximation, we will take distance travelled by the photon in each part as an average of the initial and final distance. Therefore, equating the kinematics of the situation, we have:

$$\begin{aligned}
S_1 &= \frac{D + De^{H_0 t_1}}{2} = \frac{D(1 + e^{H_0 t_1})}{2} = ct_1 \\
S_2 &= \frac{S_1 + S_1 e^{H_0 t_2}}{2} = \frac{S_1(1 + e^{H_0 t_2})}{2} = ct_2 \\
&= \frac{D}{4}(1 + e^{H_0 t_1})(1 + e^{H_0 t_2}) \\
&\approx \frac{D}{4}(2 + H_0 t_1)(2 + H_0 t_2) \\
&\approx \frac{D}{4}[4 + 2H_0(t_1 + t_2)] \\
S_2 &= D(1 + \frac{1}{2}H_o T)
\end{aligned}$$

From the first equation we also find:

$$\begin{aligned}
S_1 &= ct_1 = \frac{D(1 + e^{H_0 t_1})}{2} \\
ct_1 &= D(1 + \frac{1}{2}H_o t_1) \\
t_1 &= \frac{D}{c - \frac{1}{2}DH_0} \\
S_1 &= ct_1 = \frac{2D_c}{2c - DH_0} \\
\text{similarly, } S_2 &= \frac{2S_1 c}{2c - S_1 H_0}.
\end{aligned}$$

Joining the expressions found, we obtain:

$$S_2 = D(1+\frac{1}{2}H_0T) = \frac{2S_1c}{2c - S_1H_0}$$

$$D(1+\frac{1}{2}H_0T) = \frac{\frac{4Dc^2}{2c-DH_0}}{2c - \frac{2Dc}{2c-DH_0}H_0}$$

$$(1+\frac{1}{2}H_0T) = \frac{2c}{2c - DH_0 - DH_0} = \frac{c}{c - DH_0}$$

$$c = (1+\frac{1}{2}H_0T)(c - DH_0)$$

$$c = c - DH_0 + \frac{1}{2}cH_0T - \frac{1}{2}DH_0^2T$$

$$0 = \frac{H_0}{2}(cT - 2D - DH_0T)$$

$$H_o = \frac{cT - 2D}{DT}$$

Better approximation (considering the expansion of the Universe) would be to calculate the distance traveled as cdt rather than ct. In dt the beam travels a distance of cdt. Because the space is stretched out, the travelled distance corresponds to a smaller segment of space at t=0, smaller by a factor of $exp(H_0t)$. The distance spanned at t=0 is then

$$dr = \exp(H_0t)cdt$$

We integrate this from t=0 to t=T:

$$\int_0^{2D} dr = c\int_0^T \exp(H_0t)dt, 2D = \frac{c}{H_0}(1 - exp(-H_0T))$$

The result so far is accurate within the constraints of the model, but it is not analytically solvable for H_0. To get an estimate, we can approximate the right hand side to

$$2D = \frac{c}{H_0}(1 - 1 + H_0T - \frac{(H_0T)^2}{2}) = cT - \frac{cH_0T^2}{2}$$

Expressing H_0, we get

$$H_0 = \frac{2}{cT^2}(cT - 2D)$$

The difference between this answer and the initial estimate is $2D/cT$ which is almost unitary.

I.2.2 [3 point(s)]

Imagine that such a mirror is located in the vicinity of the star Vega. Vega was the first star outside the Solar System to be photographed and one of the first stars whose parallax (p = 0.125”) was accurately measured in 1840 by G. W. von Struve. Estimate the total duration of this H_0 measurement experiment.

Solution

From the H_0 expression found in the previous item, we find the travel time

$$T = \frac{2D}{c - DH_0} = (\frac{c}{2D} - \frac{H_0}{2})^{-1} \approx (\frac{c}{2D})^{-2} = \frac{2D}{c}$$

$$T = \frac{2 \times 8 \times 3.086 \times 10^{16}}{3 \times 10^8} = 1.65 \times 10^9 \text{s}$$

$$T \approx 52.2yr.$$

I.3 Flat Earth (5 points)

A new model of the world is gaining in popularity among some people. These people believe in the "Flat Earth" view of the world, where the Earth is not a spheroid, but rather a circle with radius $R_\oplus$. The central axis of the Earth (normal to the circle passing through its centre C) is passes through the observer's zenith. This model must at least remain consistent with the observed phenomena, as listed below:

- The value of the solar constant is $S_\odot = 1366 W/m^2$
- The Earth's central axis precesses in a circle with a period 25800 years.
- The radius of the precession circle is $23.5°$

We assume that the Earth is a perfect blackbody radiator and the Sun is sufficiently far away that all sun rays are parallel. Let us also assume that the Sun's current (initial) location is at the zenith.

I.3.1 [5 point(s)]

Determine how many years it will take for the Earth's equilibrium temperature to decrease by $1°°C$.

Solution

Assume the surface area of one side of the flat earth is A. Let the angle between the Sun and the flat Earth's center axis be θ, where θ is initially 0^o. As the Sun's rays are parallel, the power delivered to the Earth by the Sun will be $S_\odot A \cos\theta$ at any given point in time.
At equilibrium this is the energy radiated via blackbody radiation, so the equilibrium temperature T satisfies

$$S_\odot A \cos\theta = \sigma(2A)T^4,$$

where the factor 2 comes from the fact that flat Earth would radiate energy from both sides. This gives

$$T(\theta) = \sqrt[4]{\frac{S_\odot \cos\theta}{2\sigma}}.$$

We wish to find the value $\tilde{\theta}$ such that $T(\tilde{\theta}) = T(0) - \Delta T$.

$$\begin{aligned}
\sqrt[4]{\frac{S_\odot \cos\tilde{\theta}}{2\sigma}} &= \sqrt[4]{\frac{S_\odot}{2\sigma}} - \Delta T,\\
\cos\tilde{\theta} &= \left(1 - \Delta T\sqrt[4]{\frac{2\sigma}{S_\odot}}\right)^4 = \left(1 - \Delta T\sqrt[4]{\frac{2\times 5.67\times 10^{-8}}{1366}}\right)^4 = 0.9880.
\end{aligned}$$

Now we find the time it takes for the axis to make such an angle with the Sun. On the celestial sphere, let O be the center of precession, Z be the current direction of the axis, and X be the direction of the axis when it makes an angle $\tilde{\theta}$ with the sun, i.e. $\measuredangle ZCX = \tilde{\theta}$. If ϵ is the radius of precession, then $\measuredangle OCZ = \measuredangle OCX = \epsilon$.
By the spherical law of cosines on angle O of spherical triangle OXZ, we have

$$\begin{aligned}
\cos\tilde{\theta} &= \cos\epsilon\cos\epsilon + \sin\epsilon\sin\epsilon\cos(\sphericalangle O)\\
&= \cos^2\epsilon + \sin^2\epsilon\cos(\sphericalangle O)\\
\sphericalangle O &= \cos^{-1}\left(\frac{\cos\tilde{\theta} - \cos^2\epsilon}{\sin^2\epsilon}\right)\\
\Delta t &= \frac{\sphericalangle O}{2\pi}\times P = \frac{P}{2\pi}\times\cos^{-1}\left(\frac{\cos\tilde{\theta} - cos^2\epsilon}{\sin^2\epsilon}\right)\\
&= \frac{25800}{2\pi}\times\cos^{-1}\left(\frac{0.9880 - \cos^2 23.5^o}{\sin^2 23.5^o}\right)\\
&\approx 1606 \text{ yr}.
\end{aligned}$$

Therefore, the average temperature of the Earth will go down by 1^oC in just over 1600 years.

II Condensed Matter Physics

II.1 Two-site Problem (5 points)

Consider a potential representing two inequivalent wells separated by a barrier. In the limit of infinitely high barrier the two localized states have energies ε_1 and ε_2. For a finite barrier a fermion can tunnel between the states 1 and 2. Let the corresponding amplitude be τ. One can write down the Hamiltonian as

$$H = \varepsilon_1 c_1^\dagger c_1 + \varepsilon_2 c_2^\dagger c_2 - \tau(c_1^\dagger c_2 + c_2^\dagger c_1).$$

II.1.1 [2 point(s)]

Diagonalize the Hamiltonian and find its spectrum.

Solution

Introduce two new parameters ε and Δ, such that

$$\epsilon_1 = \varepsilon + \Delta, \qquad \varepsilon_2 = \varepsilon - \Delta.$$

In terms of these parameters, the Hamiltonian becomes

$$H = \varepsilon(c_1^\dagger c_1 + c_2^\dagger c_2) + \Delta(c_1^\dagger c_1 - c_2^\dagger c_2) - \tau(c_1^\dagger c_2 + c_2^\dagger c_1).$$

$c_1^\dagger c_1 + c_2^\dagger c_2$ is a total particle number operator, which is conserved and can be substituted by corresponding particle number (in our case by 1). This yields a constant number and only amounts to a uniform global shift of the energy scale - therefore we can discard it. The residual part can be compactly written as

$$H = \Psi^\dagger \mathcal{H} \Psi$$

where

$$\mathcal{H} = \Delta\sigma^z - \tau\sigma^x, \qquad \Psi = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

From $\mathcal{H} = \Delta\sigma^z - \tau\sigma^x$, the spectrum is simply $E = \pm\epsilon$, with $\epsilon = \sqrt{\tau^2 + \Delta^2}$. Using the rotation matrix

$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

such that $\cos 2\theta = \Delta/\epsilon$ and $\sin 2\theta = \tau/\epsilon$, we get

$$H = \epsilon\phi^\dagger\sigma^z\phi, \qquad \phi = S\Psi = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

II.1.2 [3 point(s)]

Imagine that at $t = 0$ the fermion was localized in the state 1. Calculate the probability to find the electron in the same state at the moment t.

Hint

Consider the transition amplitude $\langle 0|c_1(t)c_1^\dagger(0)|0\rangle$, where $c_1(t) = e^{iHt}c_1e^{-iHt}$.

Solution

To calculate the probability of finding a fermion in the same site after time t, we use the transition amplitude and the Heisenberg representation of operators.

$$\langle 0|c_1(t)c_1^\dagger(0)|0\rangle = \langle 0|c_1(t)c_1^\dagger|0\rangle.$$

Now the following manipulations are made

$$c_1(t)c_1^\dagger = e^{iHt}c_1e^{-iHt}c_1^\dagger = e^{i\epsilon t(b_1^\dagger b_1 - b_2^\dagger b_2)}(\cos\theta b_1 + \sin\theta b_2)e^{-i\epsilon t(b_1^\dagger b_1 - b_2^\dagger b_2)}c_1^\dagger.$$

Since $[b_1^\dagger b_1, b_2] = [b_2^\dagger b_2, b_1] = 0$, we have

$$c_1(t)c_1^\dagger = (\cos\theta e^{i\epsilon t b_1^\dagger b_1}b_1 e^{-i\epsilon t b_1^\dagger b_1} + \sin\theta e^{-i\epsilon t b_2^\dagger b_2}b_2 e^{i\epsilon t b_2^\dagger b_2 t})c_1^\dagger.$$

Since $b_1(t) = e^{i\epsilon b_1^\dagger b_1 t}b_1 e^{-i\epsilon b_1^\dagger b_1 t}$, the equation of motion $\frac{db_1(t)}{dt} = i[H, b_1(t)]$ gives $b_1(t) = e^{i\epsilon t}b_1$. Similarly $b_2(t) = e^{-i\epsilon t}b_2$. This way, we have

$$c_1(t)c_1^\dagger = (\cos\theta e^{i\epsilon t}b_1 + \sin\theta e^{-i\epsilon t}b_2)c_1^\dagger.$$

Expressing b_1 and b_2 back with c_1 and c_2 gives

$$c_1(t)c_1^\dagger = (\cos\theta e^{i\epsilon t}(\cos\theta c_1 - \sin\theta c_2) + \sin\theta e^{-i\epsilon t}(\sin\theta c_1 + \cos\theta c_2))c_1^\dagger.$$

Taking the average $\langle 0|...|0\rangle$, the terms that have c_2 will give zero after taking the average, since we create only one fermion at site 1 with $c_1^\dagger$ and there is nothing to destroy with c_2. Thus we have

$$\langle 0|c_1(t)c_1^\dagger|0\rangle = \cos^2\theta e^{i\epsilon t} + \sin^2\theta e^{-i\epsilon t}$$

The absolute value of this quantity gives the probability of finding a particle on site 1 at time t, when it was localized at the same site at $t = 0$.

$$P(t) = \cos^2\epsilon t + \cos^2(2\theta)\sin^2\epsilon t = \cos^2\epsilon t + \sin^2\epsilon t\frac{\Delta^2}{\sqrt{\Delta^2+\tau^2}}$$

II.2 Quantum Ising Model and Majorana Fermions (13 points)

Quantum Ising model (Also known as 1D Ising model in a transverse magnetic field) is a toy model famous for exhibiting quantum phase transition, i.e. transition not driven by temperature but instead driven by external magnetic field. We consider $T = 0$ case. The Hamiltonian of the model is:

$$\hat{\mathrm{H}}[\{\sigma\}] = J\left(-\lambda\sum_{n=1}^{N}\hat{\sigma}_n^x - \sum_{n=1}^{N-1}\hat{\sigma}_n^z\hat{\sigma}_{n+1}^z\right)^{1}. \tag{II.1}$$

$J > 0$ is the coupling constant of $z - z$ interaction and $\lambda \geq 0$ is a dimensionless parameter, that characterizes the amplitude of a transverse magnetic field $h = J\lambda$. $\hat{\sigma}_n^x$ and $\hat{\sigma}_n^z$ are the Pauli matrices for spin-$\frac{1}{2}$, each defined on a site physical site n of the chain. These Pauli matrices obey the known $SU(2)$ algebra relations when defined on the same site and commute when the site indices are different. Traditionally, we take $\hat{\sigma}_n^z$ to be a diagonal matrix, with eigenvalues of $+1$ (spin up, $|\uparrow\rangle_z$) and -1 (spin down, $|\downarrow\rangle_z$). We can also express spin up/down in x direction in the z basis as: $|\uparrow\rangle_x = |\uparrow\rangle_z + |\downarrow\rangle_z$ and $|\downarrow\rangle_x = |\uparrow\rangle_z - |\downarrow\rangle_z$. As an example, the states of this system expressed in the z basis can be written as

$$|\uparrow_1\rangle_z \otimes |\downarrow_2\rangle_z \otimes ... \otimes |\uparrow_j\rangle_z \otimes ... = |\uparrow_1, \downarrow_2, ..., \uparrow_j, ...\rangle_z \tag{II.2}$$

meaning that the first spin looks along z direction, the second in the opposite and etc. Similarly, for the x basis we have

$$|\uparrow_1\rangle_x \otimes |\downarrow_2\rangle_x \otimes ... \otimes |\uparrow_j\rangle_x \otimes ... = |\uparrow_1, \downarrow_2, ..., \uparrow_j, ...\rangle_x \tag{II.3}$$

II.2.1 [1 point(s)]

What is the global discrete symmetry of this problem? Express the unitary operator $\hat{U}$ in terms of Pauli matrices, under which

$$\hat{\sigma}_n^\alpha \to \hat{U}\hat{\sigma}_n^\alpha\hat{U}^\dagger, \qquad \text{so that} \qquad \hat{U}\hat{H}[\{\sigma\}]\hat{U}^\dagger = \hat{H}[\{\sigma\}].$$

Solution

The problem has a global $\mathbb{Z}_2$ symmetry: Flipping the sign of all $\hat{\sigma}^z$ matrices at each site n leaves the anti-commutation relations intact and the Hamiltonian stays invariant. This is equivalent to a rotation of all the spins around the x-axis by an angle π. Since $\sigma^x\sigma^z\sigma^x = -\sigma^z$ and all Pauli matrices commute if they are defined on a different sites n, the unitary operator $\hat{U}$ that corresponds to spin-flip symmetry is the following string-operator

$$\hat{U} = \prod_{i=1}^{N}\hat{\sigma_i}^x$$

II.2.2 [2 point(s)]

Argue what could be the ground state configurations of spins for $\lambda = 0$ and $\lambda = +\infty$, expressed as Eq.(II.2). What would happen if we used Eq.(II.3) instead? What are the possible values of the total spin along z direction (i.e. magnetization, interpreted as the so-called order parameter of the problem) in these limits and why should we expect a phase transition point at an intermediate value $\lambda = \lambda_c$?

Hint

The model possesses a global discrete symmetry.

[1] $\hat{\sigma}_n^z\hat{\sigma}_{n+1}^z \equiv \hat{\sigma}_n^z \otimes \hat{\sigma}_{n+1}^z$

Solution

When $\lambda = 0$, there is no transverse magnetic field in the Hamiltonian and all we are left with is $\hat{\mathrm{H}}[\{\sigma\}] = -J\sum_{n=1}^{N-1}\hat{\sigma}_n^z\hat{\sigma}_{n+1}^z$. In the ground state, all of the spins are either looking up or down in z direction- thus we have a perfect order of spins in z direction. The choice of all-up and all-down breaks the global $\mathbb{Z}_2$ symmetry and the possible magnetization is either N or $-N$. When $\lambda \to +\infty$, the transverse part of the Hamiltonian is dominating over the $z-z$ interaction and thus we have $\hat{\mathrm{H}}[\{\sigma\}] = -J\lambda\sum_{n=1}^{N}\hat{\sigma}_n^x$. In the ground state for this regime, all of the spins are looking only along x direction. This means that the global $\mathbb{Z}_2$ symmetry is not broken. On the other hand, all spins looking up in x direction translates to totally disordered spin configuration along z direction and thus there is no magnetization. The order at $\lambda = 0$ and disorder at $\lambda \to +\infty$ limits, a consequence of breakdown and recovery of $\mathbb{Z}_2$ symmetry demands a phase transition in the intermediate regime of $\lambda = \lambda_c$.

II.2.3 [3 point(s)]

To extract a precise value of λ_c, we exploit a concept of self-duality. For convenience, let us denote links that connect sites n and $n+1$ as $n+\frac{1}{2}$ and define two new matrices in the following way

$$\hat{\mu}_{n+1/2}^z = \prod_{j=1}^{n}\hat{\sigma}_j^x, \qquad \hat{\mu}_{n+1/2}^x = \hat{\sigma}_n^z\hat{\sigma}_{n+1}^z. \tag{II.4}$$

$\hat{\mu}$ matrices satisfy exactly the same algebra as $\hat{\sigma}$. Eq.(II.4) is known as the duality transformation. The action of $\hat{\mu}_{n+1/2}^z$ on spin configuration is, for instance

$$\hat{\mu}_{n+1/2}^z|\uparrow_1,\uparrow_2,\ldots,\uparrow_{n-1},\uparrow_n,\ldots,\uparrow_N\rangle_z = |\downarrow_1,\downarrow_2,\ldots,\downarrow_n,\uparrow_{n+1}\ldots,\uparrow_N\rangle_z, \tag{II.5}$$

meaning that $\hat{\mu}_{n+1/2}^z$ operator creates a domain wall in the spin configuration and thus disordering the system, hence the name - disorder operator. Invert the transformation Eq.(II.4), by expressing $\hat{\sigma}$ operators in terms of $\hat{\mu}$ operators and rewrite Eq.(II.1) Hamiltonian in terms of $\hat{\mu}$ operators, denoting it as $\hat{\mathrm{H}}[\{\mu\}]$. Let us forget for a second that initially we were working in the spin-up/down basis of $\hat{\sigma}^z$ matrices and assume that $\hat{\mathrm{H}}[\{\mu\}]$ is the starting Hamiltonian, therefore we work in spin-up/down basis of $\hat{\mu}^z$ instead. What are the possible values of the total disorder parameter in $\lambda \to +\infty$ and $\lambda = 0$ limits of $\hat{\mathrm{H}}[\{\mu\}]$ Hamiltonian? How does it compare with the same regimes for $\hat{\mathrm{H}}[\{\sigma\}]$? Extract the value of λ when the Hamiltonian maps to itself under the duality transformation, corresponding to the critical value λ_c. The Lee-Yang theorem justifies that there is only a single critical point in the model. Draw a phase diagram of $\hat{\mathrm{H}}[\{\sigma\}]$ quantum Ising model, indicating the region of order and disorder.

Solution

$$\hat{\mu}_{n-1/2}^z\hat{\mu}_{n+1/2}^z = \prod_{j=1}^{n-1}\hat{\sigma}_j^x\prod_{j=1}^{n}\hat{\sigma}_j^x = \hat{\sigma}_n^x, \qquad \prod_{j=0}^{n-1}\hat{\mu}_{n+1/2}^x = \prod_{j=0}^{n-1}\hat{\sigma}_j^z\hat{\sigma}_{j+1}^z = \hat{\sigma}_n^z.$$

The Hamiltonian written in terms of $\hat{\mu}$ matrices will be

$$\hat{\mathrm{H}}[\{\mu\}] = J\left(-\lambda\sum_{j=1}^{N}\hat{\mu}_{n-1/2}^z\hat{\mu}_{n+1/2}^z - \sum_{j=1}^{N-1}\hat{\mu}_{n+1/2}^x\right).$$

In the $\lambda = +\infty$ case, the value of the total disorder parameter is non-zero (can be N or $-N$) and in $\lambda = 0$ case the total disorder is zero. Comparing these regimes for $\hat{\mathrm{H}}[\{\sigma\}]$ and $\hat{\mathrm{H}}[\{\mu\}]$, we see that for $\lambda = 0$ the order parameter is nonzero and the disorder parameter is zero, while for $\lambda = +\infty$ the order parameter is zero and the disorder parameter is nonzero. The Hamiltonian is self-dual when $\lambda = 1$, thus $\lambda_c = 1$. For $\lambda < 1$ the system is ordered and for $\lambda > 1$ the system is disordered.

II.2.4 [3 point(s)]

The Jordan-Wigner transformation maps bosonic spin-1/2 operators onto the fermionic creation and annihilation operators in a very non-local fashion. Under the Jordan-Wigner transformation Eq.(II.1) exactly maps onto the model of a 1-dimensional p-wave superconductor (1DPS)

$$\hat{\mathrm{H}}[\{\sigma\}] \to \hat{\mathrm{H}}_{\mathrm{1DPS}} = -\lambda J \sum_{n=1}^{N} (2\hat{a}_n^\dagger \hat{a}_n - 1) - J \sum_{n=1}^{N-1} \left(\hat{a}_n^\dagger \hat{a}_{n+1} + \hat{a}_{n+1}^\dagger \hat{a}_n + \hat{a}_n^\dagger \hat{a}_{n+1}^\dagger + \hat{a}_{n+1} \hat{a}_n \right), \tag{II.6}$$

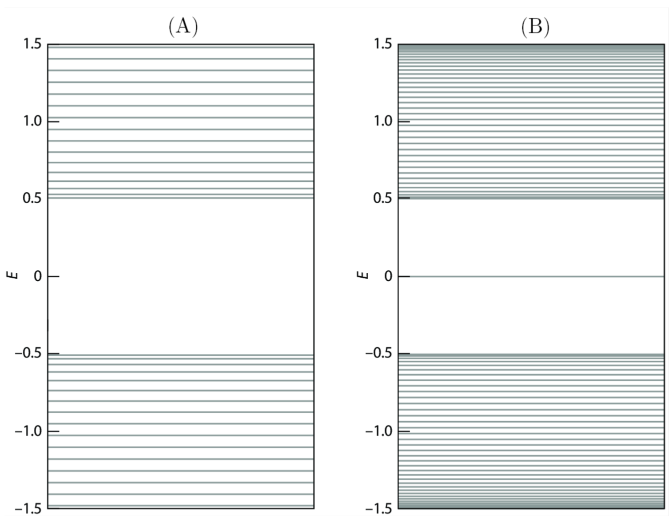
where $\hat{a}_n^\dagger$ and $\hat{a}_n$ are the creation and annihilation operators of spinless fermions at site n. These operators obey standard anti-commutation relations

$$\{\hat{a}_n^\dagger, \hat{a}_m\} = \delta_{nm}, \qquad \{\hat{a}_n, \hat{a}_m\} = 0. \tag{II.7}$$

The last two terms correspond to the superconducting coupling, creating and destroying <u>two</u> particles at a time. Due to this, the total particle number operator $\hat{Q} = \sum_{n=1}^{N} \hat{a}_n^\dagger \hat{a}_n$ does not commute with $\hat{\mathrm{H}}_{\mathrm{1DPS}}$ and the total particle number conservation is violated. However, the parity of the particle number is conserved - we either have odd or even number of particles in the system. The corresponding parity operator is

$$\hat{P} = e^{-i\pi\hat{Q}}. \tag{II.8}$$

$\hat{\mathrm{H}}_{\mathrm{1DPS}}$ model has a notorious feature in it's single-particle energy spectrum:



For $\lambda > \lambda_c$ (Same λ_c as it was in the Quantum Ising model) the spectrum of the $\hat{\mathrm{H}}_{\mathrm{1DPS}}$ is depicted on (A). However, as soon as $\lambda < \lambda_c$ condition is satisfied, depicted on (B), a mysterious energy level emerges - with energy exactly equal to zero! This corresponds to the so-called Majorana edge zero mode - a topologically protected mode, located at the left and right edges of the system. To get the essence of Majorana fermion, a simple analogy comes in handy: A complex number z can be split up as its real a and imaginary b parts, yielding $z = a + ib$. In a similar way, a creation and annihilation operator of a fermion can be represented as its "real" and "complex" parts as

$$\hat{a}_j = \frac{1}{2}\left(\hat{\zeta}_j - i\hat{\eta}_j\right), \qquad \hat{a}_j^\dagger = \frac{1}{2}\left(\hat{\zeta}_j + i\hat{\eta}_j\right), \tag{II.9}$$

where $\hat{\zeta}_j$ and $\hat{\eta}_j$ are the two Majorana fields. Derive the anti-commutation relations that the Majorana fields obey. Rewrite the Hamiltonian Eq.(II.6) and the Parity operator Eq.(II.8) in terms of the Majorana fields.

Solution

First we invert Eq.(II.9) and express $\hat{\eta}_j$ and $\hat{\zeta}_j$ in terms of $\hat{a}_j^\dagger$ and $\hat{a}_j$:

$$\hat{\zeta}_j = \hat{a}_j + \hat{a}_j^\dagger, \qquad \hat{\eta}_j = i\left(\hat{a}_j - \hat{a}_j^\dagger\right).$$

The fermionic anti-commutation relations give

$$\{\hat{\zeta}_j, \hat{\zeta}_{j'}\} = \{\hat{\eta}_j, \hat{\eta}_{j'}\} = 2\delta_{jj'}, \qquad \{\hat{\zeta}_j, \hat{\eta}_{j'}\} = 0. \tag{II.10}$$

The Hamiltonian Eq.(II.6) in terms of Majorana fields is

$$\hat{H}_{\text{1DPS}} = i\lambda J \sum_{j=1}^{N} \hat{\zeta}_j \hat{\eta}_j + iJ \sum_{j=1}^{N-1} \hat{\eta}_j \hat{\zeta}_{j+1}$$

and for the parity operator we have

$$\hat{P} = \prod_{j=1}^{N} (-i\hat{\zeta}_j \hat{\eta}_j).$$

II.2.5 [4 point(s)]

Generally, Majorana zero mode $\hat{\Psi}$ is an operator with the following properties:

$$[\hat{H}, \hat{\Psi}] = 0, \quad \{\hat{P}, \hat{\Psi}\} = 0, \quad \hat{\Psi}^\dagger \hat{\Psi}|_{N\to\infty} = 1. \tag{II.11}$$

For a zero mode to be also an edge mode, it must be localized at the boundaries of the system. Let us denote such right and left edge zero mode operator as $\hat{\Psi}_R$ and $\hat{\Psi}_L$. The matrix elements of $\hat{\Psi}_{R,L}$ must decay exponentially as we move l distance away from the corresponding boundary. If $\lambda = 0$, then $\hat{\zeta}_1$ and $\hat{\eta}_N$ do not appear in the Hamiltonian at all, they are completely isolated from the rest of the system. They also anti-commute with the parity operator and satisfy the normalization condition. Since both of them are localized at the left and right edges of the chain, they are an exact edge zero-mode operators

$$\hat{\Psi}_L(\lambda = 0) = \hat{\zeta}_1, \qquad \hat{\Psi}_R(\lambda = 0) = \hat{\eta}_N. \tag{II.12}$$

The 1DPS is in a topologically non-trivial state when $|\lambda| < \lambda_c$, therefore we need to check if the edge zero-mode operators persist as we deviate from $\lambda = 0$ point and see how does it get modified. For concreteness, let us concentrate on the left edge mode operator $\hat{\Psi}_L$ only. Develop an <u>iterative</u> method to write down the expression for $\hat{\Psi}_L^n(\lambda)$, n indicating that Eq.(II.12) has been corrected upto n order in λ. After the last possible step of the iteration, under what condition can we take $[\hat{H}_{\text{1DPS}}, \hat{\Psi}_L^{N-1}] = 0$? What about $\hat{\Psi}_R$? What happens to both modes when $|\lambda| > \lambda_c$?

Hint

Iteration method - Suppose we have two operators $\hat{A} = \hat{B} + \hat{C}$ and $\hat{Z}$, such that $[\hat{B}, \hat{Z}] = 0$, but $[\hat{C}, \hat{Z}] = \hat{D}$. Find $\hat{Z}'$, such that $[\hat{B}, \hat{Z}'] = -\hat{D}$ and add it to the initial $\hat{Z}$. Repeat.

Solution

The Hamiltonian is naturally split into two parts $\hat{H}_{\text{1DPS}} = \hat{B} + \hat{C}$

$$\hat{B} = i\lambda J \sum_{j=1}^{N} \hat{\zeta}_j \hat{\eta}_j, \quad \hat{C} = iJ \sum_{j=1}^{N-1} \hat{\eta}_j \hat{\zeta}_{j+1}.$$

$[\hat{C}, \hat{\zeta}_1] = 0$ is always valid, since $\hat{\zeta}_1$ operator is completely absent from $\hat{C}$. This way

$$[\hat{H}_{\text{1DPS}}, \hat{\zeta}_1] = [\hat{B}, \hat{\zeta}_1] = -2i\lambda J \hat{\eta}_1.$$

Introduce a new operator $\hat{o}^{(1)}$, such that $[\hat{C}, \hat{o}^{(1)}] = 2i\lambda J\hat{\eta}_1$. Such operator is $\hat{o}^{(1)} = \lambda\hat{\zeta}_2$. At this level we have $\hat{\Psi}_L^{(1)} = \hat{\zeta}_1 + \lambda\hat{\zeta}_2$. Transfer $j = 1$ term from $\hat{C}$ to $\hat{B}$ and define

$$\hat{B}^{(1)} = i\lambda J \sum_{j=1}^{N} \hat{\zeta}_j \hat{\eta}_j + iJ\hat{\eta}_1\hat{\zeta}_2, \quad \hat{C}^{(1)} = iJ \sum_{j=2}^{N-1} \hat{\eta}_j \hat{\zeta}_{j+1}.$$

Now $[\hat{H}_{\text{1DPS}}, \hat{\Psi}_L^{(1)}] = [\hat{B}^{(1)}, \hat{\Psi}_L^{(1)}] = -2i\lambda^2 J\hat{\eta}_2$. The operator $\hat{o}^{(2)}$ that gives $[\hat{C}^1, \hat{o}^{(2)}] = 2i\lambda^2 J\hat{\eta}_2$ is $\hat{o}^{(2)} = \lambda^2\hat{\zeta}_3$. Doing this iteration $N-1$ times gives the exact solution for arbitrary λ

$$\hat{\Psi}_L^{(N-1)}(\lambda) = \hat{\zeta}_1 + \lambda\hat{\zeta}_2 + \lambda^2\hat{\zeta}_3 + \ldots + \lambda^{N-1}\hat{\zeta}_N.$$

Similarly

$$\hat{\Psi}_R^{(N-1)}(\lambda) = \hat{\eta}_N + \lambda\hat{\eta}_{N-1} + \lambda^2\hat{\eta}_{N-2} + \ldots + \lambda^{N-1}\hat{\eta}_1.$$

When $|\lambda| < 1$, further we are from the boundary, the smaller is the corresponding contribution, meaning that the modes are localized at the edges of the system. $[\hat{H}_{\text{1DPS}}, \hat{\Psi}_L^{(N-1)}] \sim \lambda^N$, thus for $|\lambda| < 1$ taking a thermodynamic limit heals everything. If $|\lambda| > 1$, then non of Eq.(II.11) are satisfied.

II.3 AKLT Model (12 points)

We consider the following Hamiltonian

$$H_{\text{AKLT}} = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1} + K \sum_i \left(\vec{S}_i \cdot \vec{S}_{i+1} \right)^2 ,$$

where $\vec{S}_i = (S_i^x, S_i^y, S_i^z)$ is the vector spin-1 operator at site i.
Each site has local Hilbert space with three states: $|-1\rangle, |0\rangle, |+1\rangle$.

II.3.1 [2 point(s)]

Consider an operator $\hat{A}$ with discrete eigenvalues a_n. Define $\hat{P}^{(m)}$ as

$$\hat{P}^{(m)} = C \prod_{n \neq m} \left(\hat{A} - a_n \right) ,$$

where C is a normalization constant. Show that $\hat{P}^{(m)}$ acts as a projection operator and determine the normalization constant C.

Hint

Projector satisfies

$$\hat{P}^{(m)} |\psi_n\rangle = \begin{cases} 0 & \text{for } n \neq m, \\ |\psi_n\rangle & \text{for } n = m. \end{cases}$$

Solution

$$\begin{aligned} &\hat{P}^{(m)} |\psi_m\rangle = |\psi_m\rangle \\ &C \prod_{n \neq m} \left(\hat{A} - a_n \right) |\psi_m\rangle = |\psi_m\rangle \\ &C \prod_{n \neq m} (a_m - a_n) |\psi_m\rangle = |\psi_m\rangle \\ &C = \frac{1}{\prod_{n \neq m} (a_m - a_n)} \end{aligned}$$

II.3.2 [2 point(s)]

Consider two neighbouring sites i and $i+1$. What are the possible eigenvalues of the total spin operator $\vec{S}_{\text{tot}}^2 = \left(\vec{S}_i + \vec{S}_{i+1} \right)^2$?

Hint

How does one combine two spin-1 degrees of freedom?

Solution

$$\mathbf{1} \otimes \mathbf{1} = \mathbf{2} \oplus \mathbf{1} \oplus \mathbf{0}$$
$$\vec{S}^2_{\text{tot}} = 6, 2, 0.$$

(In general, $\vec{J}^2 = j(j+1)$.)

II.3.3 [2 point(s)]

Using the results of the first two sub-problems construct a projection operator $P^{(2)}_{i,i+1}$ that projects onto the total spin-2 subspace of the combined spin-1 degrees of freedom at sites i and $i+1$. Express it in terms of $\vec{S}_i$ and $\vec{S}_{i+1}$. What is the relationship between this projector and the AKLT Hamiltonian? What is the ground state energy of the AKLT model?

Hint

$(\vec{S})^2 = 2$ for spin 1.

Solution

$$P^{(2)}_{i,i+1} = C\left(\vec{S}^2_{\text{tot}} - 2\right)\vec{S}^2_{\text{tot}}$$
$$\vec{S}^2_{\text{tot}} = (\vec{S}_i + \vec{S}_{i+1})^2 = \vec{S}_i^2 + \vec{S}^2_{i+1} + 2\vec{S}_i \cdot \vec{S}_{i+1}$$
$$\vec{S}_i^2 = \vec{S}^2_{i+1} = 2 \implies \vec{S}^2_{\text{tot}} = 2\left(2 + \vec{S}_i \cdot \vec{S}_{i+1}\right)$$

Using $C = \frac{1}{(6-2)6} = 1/24$, we obtain

$$\begin{aligned} P^{(2)}_{i,i+1} &= \frac{1}{24}\left(2 + 2\vec{S}_i \cdot \vec{S}_{i+1}\right)\left(4 + 2\vec{S}_i \cdot \vec{S}_{i+1}\right) \\ &= \frac{1}{3} + \frac{1}{2}\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{6}\left(\vec{S}_i \cdot \vec{S}_{i+1}\right)^2. \end{aligned}$$

For $K = \frac{1}{6}$ and $J = \frac{1}{2}$,

$$H_{\text{AKLT}} = \sum_{i=0}^{N-1} P^{(2)}_{i,i+1}.$$

II.3.4 [2 point(s)]

Consider two sites i and $i+1$ and split the spin-1 degrees of freedom into two spin-1/2 degrees of freedom. The two sites combined will now have 4 spin-1/2 degrees of freedom. How can we combine these 4 degrees of freedom in order to minimize the AKLT Hamiltonian for this pair of sites? Write down the associated ground state wavefunction $|\Psi_0\rangle_{i,i+1}$ using spin-1/2 states, $|\alpha,\beta\rangle_i |\gamma,\delta\rangle_{i+1}$, where $\alpha,\beta,\gamma,\delta$ take the values $\uparrow,\downarrow$.

Hint

For instance, state $|\uparrow,\uparrow\rangle_i |\uparrow,\uparrow\rangle_{i+1}$ would mean all four spin-1/2 projections are spin-up, i.e. $S^z = +1/2$.

Solution

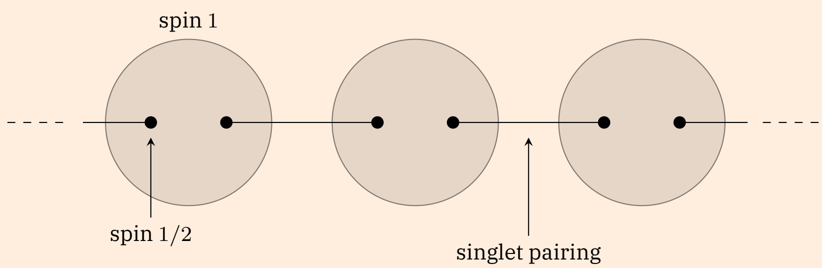
Since $\frac{\mathbf{1}}{\mathbf{2}}^{\otimes 4} = \mathbf{2} \oplus \mathbf{1}^{\oplus 3} \oplus \mathbf{0}^{\oplus 2}$ and $P^{(2)}_{i,i+1}$ projects to $\mathbf{2}$, spin-1/2 must be combined to give at most spin 1.

$$|\Psi_0\rangle_{i,i+1} = \frac{|\alpha\uparrow\rangle_i |\downarrow\beta\rangle_{i+1} - |\alpha\downarrow\rangle_i |\uparrow\beta\rangle_{i+1}}{\sqrt{2}}.$$

Whatever α, β may be, this state will have at most spin 1, since

$$\frac{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle}{\sqrt{2}}$$

is spin singlet state.



II.3.5 [2 point(s)]

Construct an operator $\hat{T}_i$ that converts from the spin-1/2 triplet basis to the spin-1 basis, e.g. $\hat{T}_i |\uparrow,\uparrow\rangle_i = |+1\rangle_i$.

Hint

One can write such operator in the form $\hat{T}_i = t^{\sigma}_{\alpha,\beta} |\sigma\rangle_i \langle\alpha,\beta|_i$, where $\alpha, \beta = \uparrow, \downarrow$ and $\sigma = -1, 0, 1$.

Solution

Let $\hat{T} = t^{\sigma}_{\alpha,\beta} |\sigma\rangle \langle\alpha,\beta|$.

$$t^{+}_{\uparrow\uparrow} = t^{-}_{\downarrow\downarrow} = 1,$$
$$t^{0}_{\uparrow\downarrow} = t^{0}_{\downarrow\uparrow} = \frac{1}{\sqrt{2}}.$$

The rest are all 0.

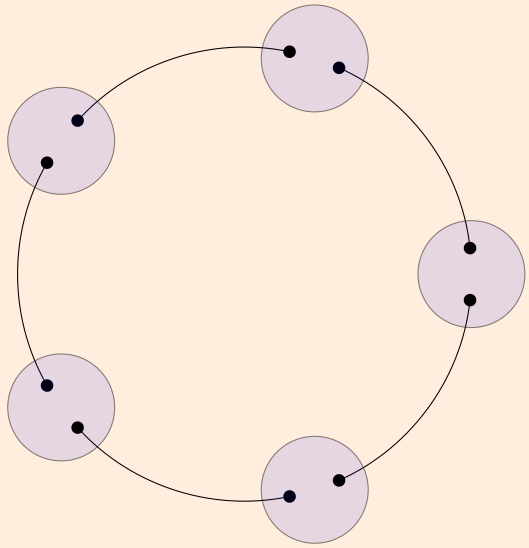
II.3.6 [2 point(s)]

Combining the results of II.3.4 and II.3.5 write down the ground state wavefunction of the AKLT model. State the difference between periodic and open boundary conditions. What is unusual about the edges in the case of open boundary conditions?

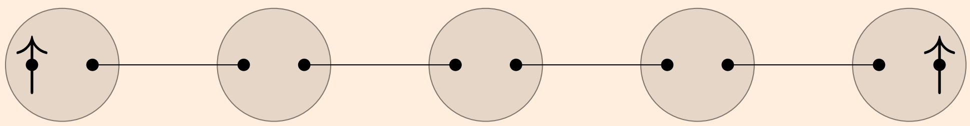
Solution

$$\begin{aligned} T_i T_{i+1} |\Psi_0\rangle_{i,i+1} &= \frac{1}{\sqrt{2}} t^{\sigma}_{\alpha\beta} t^{\sigma'}_{\alpha'\beta'} |\sigma\rangle |\sigma'\rangle \left(\langle\alpha\beta|\alpha''\uparrow\rangle \langle\alpha'\beta'|\downarrow\beta''\rangle - \langle\alpha\beta|\alpha''\downarrow\rangle \langle\alpha'\beta'|\uparrow\beta''\rangle\right) \\ &= \frac{1}{\sqrt{2}} |\sigma\rangle |\sigma'\rangle \left(t^{\sigma}_{\alpha''\uparrow} t^{\sigma'}_{\downarrow\beta''} - t^{\sigma}_{\alpha''\downarrow} t^{\sigma'}_{\uparrow\beta''}\right). \end{aligned}$$

Periodic boundary conditions:



Open boundary conditions:



For open boundary conditions we see spin-1/2 states on the edges.

III Particle Physics

III.1 Born Approximation (10 points)

III.1.1 [1 point(s)]

For a particle with mass m, the first Born approximation is defined as

$$f^{(1)}(\mathbf{k}', \mathbf{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}} V(\mathbf{x}),$$

where $V(\mathbf{x})$ is the scattering potential. Show, that for a spherically symmetric potential this simplifies to

$$f^{(1)}(\mathbf{k}', \mathbf{k}) = -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rdr\ \sin(qr)V(r).$$

The scattering is elastic.

Solution

We consider elastic scattering, when the energy is conserved and $|\mathbf{k}| = |\mathbf{k}'| \equiv k$, so we define $|\mathbf{k}-\mathbf{k}'| \equiv q = 2k\sin(\theta/2)$. In the case of the spherically symmetric potential, this simplifies to

$$\begin{aligned}
f^{(1)}(\mathbf{k}', \mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d\phi\, d(\cos(\theta'))\, r^2 dr\, e^{i|\mathbf{k}-\mathbf{k}'|r\cos(\theta')} V(r) \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}2\pi\int d(\cos(\theta'))\, r^2 dr\, e^{i|\mathbf{k}-\mathbf{k}'|r\cos(\theta')} V(r) \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}2\pi\int r^2 dr\, \left[\frac{e^{i|\mathbf{k}-\mathbf{k}'|r\cos(\theta')}}{i|\mathbf{k}-\mathbf{k}'|r}\right]_{-\pi}^{\pi} V(r) \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}2\pi\int r^2 dr\, \left[\frac{e^{i|\mathbf{k}-\mathbf{k}'|r} - e^{-i|\mathbf{k}-\mathbf{k}'|r}}{i|\mathbf{k}-\mathbf{k}'|r}\right] V(r) \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}2\pi\int r^2 dr\, \left[\frac{e^{iqr} - e^{-iqr}}{iqr}\right] V(r) \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}2\pi\int r^2 dr\, \left[\frac{2\sin(qr)}{qr}\right] V(r) \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\frac{4\pi}{q}\int rdr\ \sin(qr)V(r) \\
&= -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty rdr\ \sin(qr)V(r)
\end{aligned}$$

III.1.2 [2 point(s)]

A particle of mass m is scattered in the Yukawa potential:

$$V(r) = \frac{V_0}{r}e^{-\kappa r}.$$

Using the result above calculate the differential cross-section in the first Born approximation.

Solution

Plugging in our potential,

$$\begin{aligned} f^{(1)}(\mathbf{k}',\mathbf{k}) &= -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty r dr\ \sin(qr)V(r) \\ &= -\frac{2m}{\hbar^2}\frac{V_0}{q}\int_0^\infty r dr\ \sin(qr)\frac{e^{-\kappa r}}{r} \\ &= -\frac{2m}{\hbar^2}\frac{V_0}{q}\int_0^\infty dr\ \sin(qr)e^{-\kappa r} \\ &= -\frac{2m}{\hbar^2}\frac{V_0}{q}\mathrm{Im}\left[\int_0^\infty dr\ e^{iqr}e^{-\kappa r}\right] \end{aligned}$$

$$\begin{aligned} \int_0^\infty dr\ e^{iqr}e^{-\kappa r} &= \int_0^\infty dr\ e^{(iq-\kappa)r} \\ &= \left[\frac{e^{(iq-\kappa)r}}{iq-\kappa}\right]_0^\infty \\ &= -\frac{1}{iq-\kappa} \\ &= \frac{\kappa+iq}{\kappa^2+q^2} \end{aligned}$$

So,

$$\begin{aligned} f^{(1)}(\mathbf{k}',\mathbf{k}) &= -\frac{2m}{\hbar^2}\frac{V_0}{q}\mathrm{Im}\left[\frac{\kappa+iq}{\kappa^2+q^2}\right] \\ &= -\frac{2m}{\hbar^2}\frac{V_0}{q}\frac{q}{\kappa^2+q^2} \\ &= -\frac{2m}{\hbar^2}\frac{V_0}{\kappa^2+q^2} \\ &= -\frac{2m}{\hbar^2}\frac{V_0}{\kappa^2+4k^2\sin^2(\theta/2)} \end{aligned}$$

Cross-section is just the amplitude squared:

$$\frac{d\sigma}{d\Omega}^{(1)} = |f(\mathbf{k}',\mathbf{k})|^2 = \frac{4m^2}{\hbar^4}\frac{V_0^2}{(\kappa^2+4k^2\sin^2(\theta/2))^2}$$

III.1.3 [1 point(s)]

For what values of κ and V_0 is the Born approximation reasonable at low energies?

Solution

To check the validity of our approximation, we note that for large r the wavefunction is modulated as

$$\langle\mathbf{x}|\psi^{(+)}\rangle = \langle\mathbf{x}|\mathbf{k}\rangle - \frac{1}{4\pi}\frac{2m}{\hbar^2}\int d^3x'\ \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}V(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\ \langle\mathbf{x}'|\psi^{(+)}\rangle$$

The second part of the equation must approach zero in the scattering region (i.e. $|\mathbf{x}| \approx 0$).

$$\left| \frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3x' \, \frac{e^{ik|\mathbf{x}-\mathbf{x}'|} V(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \langle \mathbf{x}' | \mathbf{k} \rangle \right| \ll 1$$

$$\left| \frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3x' \, \frac{e^{ik|\mathbf{x}'|} V(\mathbf{x}')}{|\mathbf{x}'|} \frac{e^{i\mathbf{k}\cdot\mathbf{x}'}}{(2\pi)^{3/2}} \right| \ll 1$$

$$\left| \frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3x' \, \frac{e^{ikx'} V(x')}{x'} \frac{e^{ikx' \cos\theta}}{(2\pi)^{3/2}} \right| \ll 1$$

$$\left| \frac{1}{4\pi} \frac{2m}{\hbar^2} 4\pi \int (x')^2 dx' \, \frac{e^{ikx'} V(x')}{x'} \frac{\sin(kx')}{(2\pi)^{3/2} kx'} \right| \ll 1$$

$$\left| \frac{2m}{\hbar^2} \frac{V_0}{(2\pi)^{3/2}} \int (x')^2 dx' \, \frac{e^{ikx'}}{x'} \frac{e^{-\kappa x'}}{x'} \frac{\sin(kx')}{kx'} \right| \ll 1$$

$$\left| \frac{2m}{\hbar^2} \frac{V_0}{(2\pi)^{3/2}} \int dx' \, e^{ikx'} e^{-\kappa x'} \frac{\sin(kx')}{kx'} \right| \ll 1$$

In the approximation of small k,

$$\left| e^{ikx'} \right| = 1 = \left| \frac{\sin(kx')}{kx'} \right|$$

So, we're left with

$$\left| \frac{2m}{\hbar^2} \frac{V_0}{(2\pi)^{3/2}} \int dx' \, e^{-\kappa x'} \right| \ll 1$$

$$\left| \frac{2m}{\hbar^2} \frac{V_0}{(2\pi)^{3/2}} \frac{1}{\kappa} \right| \ll 1$$

$$\left| \frac{2m}{\hbar^2} \frac{V_0}{\kappa} \right| \ll 1$$

where we have removed the insignificant factor $(2\pi)^{3/2}$.

III.1.4 [1 point(s)]

In the limit $\kappa \to 0$ Yukawa potential transforms into Coulomb interaction. Show that the cross-section (or rather, the first Born approximation) describes Rutherford scattering in this limit.

Solution

We set $\kappa = 0$ and $V_0 = ZZ'e^2$, so one obtains

$$\begin{aligned} \frac{d\sigma}{d\Omega}^{(1)} &= \frac{4m^2}{\hbar^4} \frac{V_0^2}{(\kappa^2 + 4k^2 \sin^2(\theta/2))^2} \\ &= \frac{4m^2}{\hbar^4} \frac{\left(ZZ'e^2\right)^2}{(4k^2 \sin^2(\theta/2))^2} \\ &= \frac{4m^2}{\hbar^4} \frac{\left(ZZ'e^2\right)^2}{16k^4 \sin^4(\theta/2)} \end{aligned}$$

III.1.5 [2 point(s)]

The second Born amplitude is defined as

$$f^{(2)}(\mathbf{k}', \mathbf{k}) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} (2\pi)^3 \left\langle \mathbf{k}' \middle| V \frac{1}{E - H_0 + i\varepsilon} V \middle| \mathbf{k} \right\rangle.$$

Show that the forward scattering amplitude for the Yukawa potential is given by

$$f^{(2)}(\mathbf{k},\mathbf{k}) = -4\pi\left(\frac{2m}{\hbar^2}\right)^2 \frac{V_0^2}{(2\pi)^3}4\pi\int_0^\infty \frac{\tilde{k}^2 d\tilde{k}}{(k^2-\tilde{k}^2+i\epsilon)(\kappa^2+(k-\tilde{k})^2)(\kappa^2+(k+\tilde{k})^2)}$$

Solution

The forward scattering condition just sets $\mathbf{k}' = \mathbf{k}$. So,

$$\begin{aligned}
f^{(2)}(\mathbf{k},\mathbf{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\left\langle \mathbf{k}\middle| V\frac{1}{E-H_0+i\varepsilon}V\middle|\mathbf{k}\right\rangle \\
&= -\frac{1}{4\pi}\frac{2m}{\hbar^2}(2\pi)^3\int d^3\tilde{k}\frac{1}{E-\hbar^2\tilde{k}^2/2m+i\varepsilon}\langle\mathbf{k}|V|\tilde{\mathbf{k}}\rangle\langle\tilde{\mathbf{k}}|V|\mathbf{k}\rangle \\
&= -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)^2(2\pi)^3\int d^3\tilde{k}\frac{1}{k^2-\tilde{k}^2+i\epsilon}|\langle\mathbf{k}|V|\tilde{\mathbf{k}}\rangle|^2
\end{aligned}$$

We have used the hermitian property of V operator. Next, we calculate the matrix product:

$$\begin{aligned}
\langle\mathbf{k}|V|\tilde{\mathbf{k}}\rangle &= \int d^3x\,\langle\mathbf{k}|V|\mathbf{x}\rangle\langle\mathbf{x}|\tilde{\mathbf{k}}\rangle \\
&= V_0\int d^3x\frac{e^{-\kappa r}}{r}\langle\mathbf{k}|\mathbf{x}\rangle\langle\mathbf{x}|\tilde{\mathbf{k}}\rangle \\
&= V_0\int d^3x\frac{e^{-\kappa r}}{r}\frac{e^{i(\tilde{\mathbf{k}}-\mathbf{k})\cdot\mathbf{x}}}{(2\pi)^3} \\
&= 2\pi V_0\int d(\cos\theta)r^2dr\frac{e^{-\kappa r}}{r}\frac{e^{i|\tilde{\mathbf{k}}-\mathbf{k}|r\cos\theta}}{(2\pi)^3} \\
&= 4\pi V_0\int r^2dr\frac{e^{-\kappa r}}{r}\frac{\sin(|\tilde{\mathbf{k}}-\mathbf{k}|r)}{(2\pi)^3|\tilde{\mathbf{k}}-\mathbf{k}|r} \\
&= 4\pi\frac{V_0}{(2\pi)^3q}\int dre^{-\kappa r}\sin(qr) \\
&= 4\pi\frac{V_0}{(2\pi)^3q}\mathrm{Im}\left[\int_0^\infty dre^{(iq-\kappa)r}\right] \\
&= 4\pi\frac{V_0}{(2\pi)^3}\frac{1}{\kappa^2+q^2} \\
&= 4\pi\frac{V_0}{(2\pi)^3}\frac{1}{\kappa^2+k^2+\tilde{k}^2+2k\tilde{k}\cos(\tilde{\theta})}
\end{aligned}$$

Plugging this back to $f^{(2)}$,we have

$$\begin{aligned}
f^{(2)}(\mathbf{k},\mathbf{k}) &= -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)^2(2\pi)^3\int d^3\tilde{k}\frac{1}{k^2-\tilde{k}^2+i\epsilon}|\langle\mathbf{k}|V|\tilde{\mathbf{k}}\rangle|^2 \\
&= -\frac{1}{4\pi}\left(\frac{2m}{\hbar^2}\right)^2(2\pi)^3\int d^3\tilde{k}\frac{1}{k^2-\tilde{k}^2+i\epsilon}\left(4\pi\frac{V_0}{(2\pi)^3}\frac{1}{\kappa^2+k^2+\tilde{k}^2+2k\tilde{k}\cos(\tilde{\theta})}\right)^2 \\
&= -4\pi\left(\frac{2m}{\hbar^2}\right)^2\frac{V_0^2}{(2\pi)^3}\int d^3\tilde{k}\frac{1}{k^2-\tilde{k}^2+i\epsilon}\left(\frac{1}{\kappa^2+k^2+\tilde{k}^2+2k\tilde{k}\cos(\tilde{\theta})}\right)^2 \\
&= -4\pi\left(\frac{2m}{\hbar^2}\right)^2\frac{V_0^2}{(2\pi)^3}2\pi\int d(\cos\tilde{\theta})\tilde{k}^2d\tilde{k}\frac{1}{k^2-\tilde{k}^2+i\epsilon}\left(\frac{1}{\kappa^2+k^2+\tilde{k}^2+2k\tilde{k}\cos(\tilde{\theta})}\right)^2
\end{aligned}$$

We can already integrate the cosine:

$$\begin{aligned}
\int_{-1}^1 d(\cos\tilde{\theta})\left(\frac{1}{\kappa^2+k^2+\tilde{k}^2+2k\tilde{k}\cos(\tilde{\theta})}\right)^2 &= \frac{1}{2k\tilde{k}}\left[\frac{1}{\kappa^2+k^2+\tilde{k}^2-2k\tilde{k}}-\frac{1}{\kappa^2+k^2+\tilde{k}^2+2k\tilde{k}}\right] \\
&= \frac{1}{2k\tilde{k}}\left[\frac{1}{\kappa^2+(k-\tilde{k})^2}-\frac{1}{\kappa^2+(k+\tilde{k})^2}\right] \\
&= \frac{2}{(\kappa^2+(k-\tilde{k})^2)(\kappa^2+(k+\tilde{k})^2)}
\end{aligned}$$

So, our equation now is:

$$f^{(2)}(\mathbf{k},\mathbf{k}) = -4\pi\left(\frac{2m}{\hbar^2}\right)^2 \frac{V_0^2}{(2\pi)^3}4\pi\int_0^\infty \frac{\tilde{k}^2 d\tilde{k}}{(k^2-\tilde{k}^2+i\epsilon)(\kappa^2+(k-\tilde{k})^2)(\kappa^2+(k+\tilde{k})^2)}$$

III.1.6 [2 point(s)]

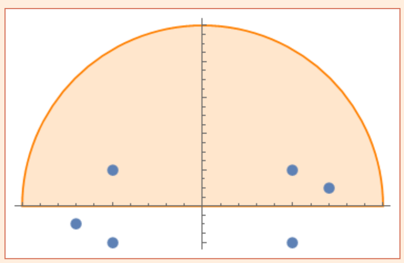
Identify all the poles of the integrand in the above result and integrate it over all $\tilde{k}$ to obtain

$$f^{(2)}(\mathbf{k},\mathbf{k}) = \left(\frac{2m}{\hbar^2}\right)^2 \frac{V_0^2}{2\kappa^2(\kappa-2ik)}.$$

Solution

$$\begin{aligned} f^{(2)}(\mathbf{k},\mathbf{k}) &= -4\pi\left(\frac{2m}{\hbar^2}\right)^2 \frac{V_0^2}{(2\pi)^3}4\pi\int_0^\infty \frac{\tilde{k}^2 d\tilde{k}}{(k^2-\tilde{k}^2+i\epsilon)(\kappa^2+(k-\tilde{k})^2)(\kappa^2+(k+\tilde{k})^2)} \\ &= -4\pi\left(\frac{2m}{\hbar^2}\right)^2 \frac{V_0^2}{(2\pi)^3}\frac{4\pi}{2}\int_{-\infty}^\infty \frac{\tilde{k}^2 d\tilde{k}}{(k^2-\tilde{k}^2+i\epsilon)(\kappa^2+(k-\tilde{k})^2)(\kappa^2+(k+\tilde{k})^2)} \end{aligned}$$

We have symmetrically doubled the limits of the integral in the last step. We can do this, since the integrand is an even function of the integration variable. The function has six poles in the complex plane:



Corresponding points are:

$$\begin{aligned} \alpha_1 &= k+i\kappa \\ \alpha_2 &= k-i\kappa \\ \alpha_3 &= -k+i\kappa \\ \alpha_4 &= -k-i\kappa \\ \alpha_5 &= \sqrt{k^2+i\epsilon} \\ \alpha_6 &= -\sqrt{k^2+i\epsilon} \end{aligned}$$

Only half of these zeros (namely, $\alpha_1, \alpha_3, \alpha_5$) are on the upper part of the plane. We can integrate on the upper half-circle, having a guarantee that the circular part gives zero after $R\to\infty$, because the integrand goes as $1/\tilde{k}^3$ and becomes zero at very far distances.
So, using the residue theorem,

$$\begin{aligned} f^{(2)}(\mathbf{k},\mathbf{k}) &= -4\pi\left(\frac{2m}{\hbar^2}\right)^2 \frac{V_0^2}{(2\pi)^3}\frac{4\pi}{2}\left[2i\pi\left(\text{Res}\,(f,\alpha_1)+\text{Res}\,(f,\alpha_3)+\text{Res}\,(f,\alpha_5)\right)\right] \\ &= -4\pi\left(\frac{2m}{\hbar^2}\right)^2 \frac{V_0^2}{(2\pi)^3}\frac{4\pi}{2}\left[-\frac{\pi}{2\kappa^2(\kappa-2ik)}\right] \\ &= \left(\frac{2m}{\hbar^2}\right)^2 \frac{V_0^2}{2\kappa^2(\kappa-2ik)} \end{aligned}$$

III.1.7 [1 point(s)]

The optical theorem relates the full cross-section to the imaginary part of the forward scattering amplitude. State the optical theorem and check that it holds for the Yukawa potential (the first terms in powers of V_0). Why is the second Born approximation needed for this?

Solution

The optical theorem states:

$$\begin{aligned}\operatorname{Im}\left[f(\mathbf{k},\mathbf{k})\right] &= \frac{k}{4\pi}\sigma_{tot} \\ &= \frac{k}{4\pi}\int d\Omega \frac{d\sigma}{d\Omega} \\ &= \frac{k}{4\pi}\int d\Omega \left|f(\mathbf{k}',\mathbf{k})\right|^2\end{aligned}$$

The first born amplitude contains V_0 linearly, the second one contains it squared and so on. If we need to check term by term, we need to compare the imaginary part of the *second* amplitude to the total cross section calculated using the *first* one. In other words,

$$\begin{aligned}\operatorname{Im}\left[f^{(2)}(\mathbf{k},\mathbf{k})\right] &= \frac{k}{4\pi}\int d\Omega \left|f^{(1)}(\mathbf{k}',\mathbf{k})\right|^2 \\ &= \frac{k}{4\pi}\int d\phi\, d(\cos\theta)\frac{4m^2}{\hbar^4}\frac{V_0^2}{(\kappa^2+2k^2-2k^2\cos(\theta))^2} \\ &= \frac{k}{4\pi}4\pi\frac{4m^2}{\hbar^4}\frac{V_0^2}{\kappa^2(\kappa^2+4k^2)} \\ &= k\frac{4m^2}{\hbar^4}\frac{V_0^2}{\kappa^2(\kappa^2+4k^2)}\end{aligned}$$

Comparing this to the $f^{(2)}$ above, we see that the optical theorem is satisfied.

III.2 The Higgs Mechanism (10 points)

Consider the following Lagrangian:

$$\mathcal{L} = (D^\mu\Phi)^\dagger D_\mu\Phi - \mu^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2$$

where

- $\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix}$ is an $SU(2)$ doublet;
- $D_\mu = \partial_\mu + ig\frac{\tau^a}{2}W_\mu^a$ is the covariant derivative;
- τ^a denote the Pauli matrices (see the Appendix below), $a = 1, 2, 3$;
- W_μ^a are vector bosons;
- g is a coupling constant.

III.2.1 [1 point(s)]

Under the local $SU(2)$ transformations

$$\Phi \to e^{i\alpha^a(x)\frac{\tau^a}{2}}\Phi,$$

the vector fields transform as

$$W_\mu^a \to W_\mu^a - \frac{1}{g}\partial_\mu\alpha^a(x) - \epsilon^{abc}\alpha^b(x)W_\mu^c.$$

ϵ^{abc} is the totally-antisymmetric symbol with $\epsilon^{123} = 1$.

Show that a mass term for the vector bosons breaks the gauge invariance of the Lagrangian.

Solution

Let's consider a mass-term for W_μ^a fields:

$$\mathcal{L}_{M_W} = -\frac{M_{ab}}{2}W_\mu^a(W^b)^\mu$$

The transformation will be:

$$\begin{aligned}
\mathcal{L}_{M_W} &\to -\frac{M_{ab}}{2}\left(W_\mu^a - \frac{1}{g}\partial_\mu\alpha^a(x) - \epsilon^{amn}\alpha^m(x)W_\mu^n\right)\left((W^b)^\mu - \frac{1}{g}\partial^\mu\alpha^b(x) - \epsilon^{bij}\alpha^i(x)(W^j)^\mu\right) \\
&= -\frac{M_{ab}}{2}W_\mu^a(W^b)^\mu + \frac{M_{ab}}{2}\left(W_\mu^a\frac{1}{g}\partial^\mu\alpha^b + \frac{1}{g}\partial_\mu\alpha^a(W^b)^\mu + W_\mu^a\epsilon^{bij}\alpha^i(W^j)^\mu + \epsilon^{amn}\alpha^m W_\mu^n(W^b)^\mu\right) \\
&= -\frac{M_{ab}}{2}W_\mu^a(W^b)^\mu + \frac{M_{ab}}{2g}\partial^\mu\alpha^{(a}W_\mu^{b)} + \frac{M_{ab}}{2}W_\mu^{(a}\epsilon^{b)ij}\alpha^i(W^j)^\mu
\end{aligned}$$

where (a, b) denote the symmetrization in indices a, b. We note, that the additional term has a structure $M_{ab}S^{ab}$, where S^{ab} is symmetric. The only way that this expression is identically zero, is when M_{ab} is anti-symmetric. But mass matrices are defined to be symmetric and positive definite. So, an anti-symmetric M_{ab} won't do.

III.2.2 [1 point(s)]

We assume $\lambda > 0$, so that the potential

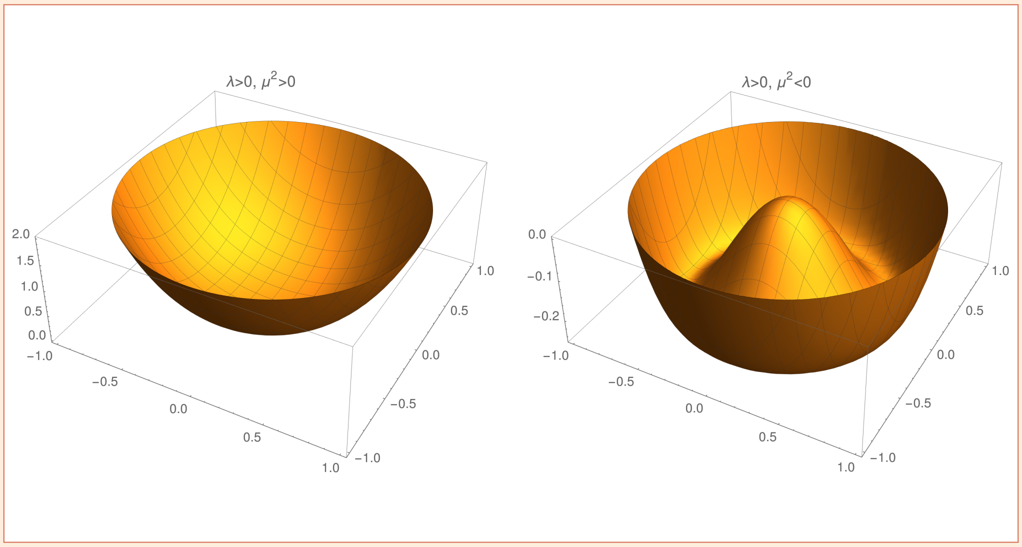
$$V = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$$

is bounded from below.

Which case describes a theory with spontaneous symmetry breaking: $\mu^2 > 0$ or $\mu^2 < 0$?

Solution

For the case that Φ is a complex number (or a real two-component matrix), the plots are:



where horizontal axes are either Φ^* and Φ, or Φ_1 and Φ_2.
It's clear that $\mu^2 > 0$ case does not break the symmetry, but $\mu^2 < 0$ case does.

III.2.3 [1 point(s)]

What conditions must the fields $\Phi, \Phi^\dagger$ satisfy in order to minimize V?

Solution

First of all, notice, that all components of Φ enter in the expression of V only as a product $\Phi^\dagger \Phi$. So, any condition will be ultimately imposed on $\Phi^\dagger \Phi$.

$$V(\Phi^\dagger \Phi) = \mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2$$

The condition of the minimum is

$$\left.\frac{\partial V}{\partial(\Phi^\dagger\Phi)}\right|_{min} = 0$$
$$\mu^2 + 2\lambda\Phi^\dagger\Phi = 0$$
$$\Phi^\dagger\Phi = \frac{-\mu^2}{2\lambda}$$
$$\text{By definition,}\quad \Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{pmatrix}$$
$$\text{So,}\;\; \phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = \frac{-\mu^2}{\lambda} \equiv v^2$$

III.2.4 [1 point(s)]

For the ground state we choose

$$\Phi_0 = \begin{pmatrix}0\\ v\end{pmatrix}.$$

In other words, we set $\phi_1 = \phi_2 = \phi_3 = 0$ and $\phi_4 = v = \text{const}$. Why are we allowed to do this? What is the value for the constant v?

Solution

The result of the previous part gives that the fields ϕ_i satisfy the following constraint:

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = \frac{-\mu^2}{\lambda} \equiv v^2$$

i.e. they are located on a 3-sphere. We can choose any point for this 3-sphere as a ground state and expand the general fields around it.
The constant $v = \sqrt{-\mu^2/\lambda}$. Note, that since $\mu^2 < 0$, v is real.

III.2.5 [1 point(s)]

We expand the fields around Φ_0:

$$\Phi = \Phi_0 + \Delta\Phi = \begin{pmatrix}0\\ v\end{pmatrix} + \begin{pmatrix}\Delta\phi_1(x) + i\Delta\phi_2(x)\\ \Delta\phi_3(x) + i\Delta\phi_4(x)\end{pmatrix} = \begin{pmatrix}\Delta\phi_1(x) + i\Delta\phi_2(x)\\ v + \Delta\phi_3(x) + i\Delta\phi_4(x)\end{pmatrix}$$

Show that this is equivalent to the infinitesimal transformation

$$\Phi = \frac{1}{\sqrt{2}} e^{i\frac{\theta^a(x)}{v}\tau^a}\begin{pmatrix}0\\ v + h(x)\end{pmatrix}.$$

How are the fields $\Delta\phi_1, \Delta\phi_2, \Delta\phi_3, \Delta\phi_4$ given in terms of $\theta_1, \theta_2, \theta_3, h$?

Solution

We consider infinitesimal fields, so

$$\begin{aligned}
e^{i\frac{\theta^a(x)}{v}\tau^a} &\approx \left(1 + i\frac{\theta^a(x)}{v}\tau^a\right) \\
&= \begin{pmatrix} 1 + i\theta_3/v & i(\theta_1 - i\theta_2)/v \\ i(\theta_1 + i\theta_2)/v & 1 - i\theta_3/v \end{pmatrix}
\end{aligned}$$

Now,

$$\begin{aligned}
\frac{1}{\sqrt{2}}e^{i\frac{\theta^a}{v}\tau^a}\begin{pmatrix} 0 \\ v+h \end{pmatrix} &\approx \frac{1}{\sqrt{2}}\begin{pmatrix} 1 + i\theta_3/v & i(\theta_1 - i\theta_2)/v \\ i(\theta_1 + i\theta_2)/v & 1 - i\theta_3/v \end{pmatrix}\begin{pmatrix} 0 \\ v+h \end{pmatrix} \\
&= \frac{1}{\sqrt{2}}\begin{pmatrix} \theta_2 + i\theta_1 + (\theta_2 + i\theta_1)h/v \\ v + h - i\theta_3 - i\theta_3 h/v \end{pmatrix}
\end{aligned}$$

Small fields means that products $\theta_i(x)h(x)$ are also infinitesimally small, so finally,

$$\Phi \approx \frac{1}{\sqrt{2}}\begin{pmatrix} \theta_2(x) + i\theta_1(x) \\ v + h(x) - i\theta_3(x) \end{pmatrix}$$

This is equivalent to the perturbation given above, with

$$\Delta\phi_1(x) = \theta_2(x), \quad \Delta\phi_2(x) = \theta_1(x), \quad \Delta\phi_3(x) = h(x), \quad \Delta\phi_4(x) = -\theta_3(x).$$

III.2.6 [1 point(s)]

Consider the kinetic part of the Lagrangian:

$$\mathcal{L}_{kin} = (D^\mu\Phi)^\dagger(D_\mu\Phi)$$

Show that inserting

$$\Phi = \frac{1}{\sqrt{2}}e^{i\frac{\theta^a(x)}{v}\tau^a}\begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}.$$

into $\mathcal{L}_{kin}$ gives

$$\begin{aligned}
\mathcal{L}_{kin} = &\frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{1}{2}(\partial^\mu\theta_1)(\partial_\mu\theta_1) + \frac{1}{2}(\partial^\mu\theta_2)(\partial_\mu\theta_2) + \frac{1}{2}(\partial^\mu\theta_3)(\partial_\mu\theta_3) \\
&+ \frac{g}{2}W^1_\mu(h\partial^\mu\theta_1 + v\partial^\mu\theta_1 - \theta_1\partial^\mu h + \theta_3\partial^\mu\theta_2 - \theta_2\partial^\mu\theta_3) \\
&+ \frac{g}{2}W^2_\mu(h\partial^\mu\theta_2 + v\partial^\mu\theta_2 - \theta_2\partial^\mu h + \theta_1\partial^\mu\theta_3 - \theta_3\partial^\mu\theta_1) \\
&+ \frac{g}{2}W^3_\mu(h\partial^\mu\theta_3 + v\partial^\mu\theta_3 - \theta_3\partial^\mu h + \theta_2\partial^\mu\theta_1 - \theta_1\partial^\mu\theta_2) \\
&+ \frac{g^2}{8}\left((W^1_\mu)^2 + (W^2_\mu)^2 + (W^3_\mu)^2\right)(v^2 + 2vh + h^2 + \theta_1^2 + \theta_2^2 + \theta_3^2)
\end{aligned}$$

Solution

To calculate this, we're going to need

$$
\begin{aligned}
D_\mu \Phi &= \left(\partial_\mu + ig\frac{\tau^a}{2}W^a_\mu\right)\frac{1}{\sqrt{2}}\begin{pmatrix}\theta_2(x)+i\theta_1(x)\\ v+h(x)-i\theta_3(x)\end{pmatrix}\\
&= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}\partial_\mu\theta_2(x)+i\partial_\mu\theta_1(x)\\ \partial_\mu h(x)-i\partial_\mu\theta_3(x)\end{pmatrix} + ig\frac{\tau^a}{2}W^a_\mu\begin{pmatrix}\theta_2(x)+i\theta_1(x)\\ v+h(x)-i\theta_3(x)\end{pmatrix}\right]\\
&= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}\partial_\mu\theta_2(x)+i\partial_\mu\theta_1(x)\\ \partial_\mu h(x)-i\partial_\mu\theta_3(x)\end{pmatrix} + \frac{ig}{2}\begin{pmatrix}W^3_\mu & W^1_\mu - iW^2_\mu\\ W^1_\mu+iW^2_\mu & -W^3_\mu\end{pmatrix}\begin{pmatrix}\theta_2(x)+i\theta_1(x)\\ v+h(x)-i\theta_3(x)\end{pmatrix}\right]\\
&= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}\partial_\mu\theta_2(x)+i\partial_\mu\theta_1(x)\\ \partial_\mu h(x)-i\partial_\mu\theta_3(x)\end{pmatrix} + \frac{ig}{2}\begin{pmatrix}W^3_\mu(i\theta_1(x)+\theta_2(x)) + (W^1_\mu - iW^2_\mu)(v+h(x)+i\theta_3(x))\\ (W^1_\mu+iW^2_\mu)(i\theta_1(x)+\theta_2(x)) - W^3_\mu(v+h(x)+i\theta_3(x))\end{pmatrix}\right]
\end{aligned}
$$

A bit too overwhelming... Let's denote

$$
\frac{1}{\sqrt{2}}\begin{pmatrix}\theta_2(x)+i\theta_1(x)\\ v+h(x)-i\theta_3(x)\end{pmatrix} \equiv \frac{1}{\sqrt{2}}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}
$$

So,

$$
\begin{aligned}
D_\mu\Phi &= \frac{1}{\sqrt{2}}\left[\begin{pmatrix}\partial_\mu\Phi_1\\ \partial_\mu\Phi_2\end{pmatrix} + \frac{ig}{2}\begin{pmatrix}W^3_\mu\Phi_1 + (W^1_\mu - iW^2_\mu)\Phi_2\\ (W^1_\mu + iW^2_\mu)\Phi_1 - W^3_\mu\Phi_2\end{pmatrix}\right]\\
&= \frac{1}{\sqrt{2}}\begin{pmatrix}(\partial_\mu + \frac{ig}{2}W^3_\mu)\Phi_1 + \frac{ig}{2}(W^1_\mu - iW^2_\mu)\Phi_2\\ (\partial_\mu - \frac{ig}{2}W^3_\mu)\Phi_2 + \frac{ig}{2}(W^1_\mu + iW^2_\mu)\Phi_1\end{pmatrix}\\
&\xrightarrow[\frac{g}{2}W^a_\mu \equiv w^a_\mu]{\text{redefine:}} = \frac{1}{\sqrt{2}}\begin{pmatrix}(\partial_\mu + iw^3_\mu)\Phi_1 + (w^2_\mu + iw^1_\mu)\Phi_2\\ (\partial_\mu - iw^3_\mu)\Phi_2 + (-w^2_\mu + iw^1_\mu)\Phi_1\end{pmatrix}
\end{aligned}
$$

Now,

$$
\begin{aligned}
\left(\partial_\mu + iw^3_\mu\right)\left(\theta_2(x)+i\theta_1(x)\right) &= \left(\partial_\mu\theta_2 - w^3_\mu\theta_1\right) + i\left(\partial_\mu\theta_1 + w^3_\mu\theta_2\right)\\
\left(w^2_\mu + iw^1_\mu\right)\left(v+h(x)-i\theta_3(x)\right) &= \left(w^2_\mu(v+h) + w^1_\mu\theta_3\right) + i\left(w^1_\mu(v+h) - w^2_\mu\theta_3\right)\\
\left(\partial_\mu - iw^3_\mu\right)\left(v+h(x)-i\theta_3(x)\right) &= \left(\partial_\mu h - w^3_\mu\theta_3\right) - i\left(\partial_\mu\theta_3 + w^3_\mu(v+h)\right)\\
\left(-w^2_\mu + iw^1_\mu\right)\left(\theta_2(x)+i\theta_1(x)\right) &= -\left(w^2_\mu\theta_2 + w^1_\mu\theta_1\right) + i\left(w^1_\mu\theta_2 - w^2_\mu\theta_1\right)
\end{aligned}
$$

So, we can rewrite

$$
D_\mu\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}\xi_1 + i\xi_2\\ \xi_3 + i\xi_4\end{pmatrix}
$$

where

$$
\begin{aligned}
\xi_1 &= \partial_\mu\theta_2 - w^3_\mu\theta_1 + w^2_\mu(v+h) + w^1_\mu\theta_3\\
\xi_2 &= \partial_\mu\theta_1 + w^3_\mu\theta_2 + w^1_\mu(v+h) - w^2_\mu\theta_3\\
\xi_3 &= \partial_\mu h - w^3_\mu\theta_3 - w^2_\mu\theta_2 - w^1_\mu\theta_1\\
\xi_4 &= -\partial_\mu\theta_3 - w^3_\mu(v+h) + w^1_\mu\theta_2 - w^2_\mu\theta_1
\end{aligned}
$$

And finally, $(D^\mu\Phi)^\dagger(D_\mu\Phi) = (\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)/2$, so,

$$
\begin{aligned}
\mathcal{L}_{kin} = &\frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{1}{2}(\partial^\mu\theta_1)(\partial_\mu\theta_1) + \frac{1}{2}(\partial^\mu\theta_2)(\partial_\mu\theta_2) + \frac{1}{2}(\partial^\mu\theta_3)(\partial_\mu\theta_3)\\
&+ w^1_\mu\left(h\partial^\mu\theta_1 + v\partial^\mu\theta_1 - \theta_1\partial^\mu h + \theta_3\partial^\mu\theta_2 - \theta_2\partial^\mu\theta_3\right)\\
&+ w^2_\mu\left(h\partial^\mu\theta_2 + v\partial^\mu\theta_2 - \theta_2\partial^\mu h + \theta_1\partial^\mu\theta_3 - \theta_3\partial^\mu\theta_1\right)\\
&+ w^3_\mu\left(h\partial^\mu\theta_3 + v\partial^\mu\theta_3 - \theta_3\partial^\mu h + \theta_2\partial^\mu\theta_1 - \theta_1\partial^\mu\theta_2\right)\\
&+ \frac{1}{2}\left(\left(w^1_\mu\right)^2 + \left(w^2_\mu\right)^2 + \left(w^3_\mu\right)^2\right)\left(v^2 + 2vh + h^2 + \theta_1^2 + \theta_2^2 + \theta_3^2\right)
\end{aligned}
$$

In terms of W^a_μ :

$$
\begin{aligned}
\mathcal{L}_{kin} = &\frac{1}{2}(\partial^\mu h)(\partial_\mu h) + \frac{1}{2}(\partial^\mu\theta_1)(\partial_\mu\theta_1) + \frac{1}{2}(\partial^\mu\theta_2)(\partial_\mu\theta_2) + \frac{1}{2}(\partial^\mu\theta_3)(\partial_\mu\theta_3)\\
&+ \frac{g}{2}W^1_\mu\left(h\partial^\mu\theta_1 + v\partial^\mu\theta_1 - \theta_1\partial^\mu h + \theta_3\partial^\mu\theta_2 - \theta_2\partial^\mu\theta_3\right)\\
&+ \frac{g}{2}W^2_\mu\left(h\partial^\mu\theta_2 + v\partial^\mu\theta_2 - \theta_2\partial^\mu h + \theta_1\partial^\mu\theta_3 - \theta_3\partial^\mu\theta_1\right)\\
&+ \frac{g}{2}W^3_\mu\left(h\partial^\mu\theta_3 + v\partial^\mu\theta_3 - \theta_3\partial^\mu h + \theta_2\partial^\mu\theta_1 - \theta_1\partial^\mu\theta_2\right)\\
&+ \frac{g^2}{8}\left(\left(W^1_\mu\right)^2 + \left(W^2_\mu\right)^2 + \left(W^3_\mu\right)^2\right)\left(v^2 + 2vh + h^2 + \theta_1^2 + \theta_2^2 + \theta_3^2\right)
\end{aligned}
$$

III.2.7 [1 point(s)]

Consider the potential part of the Lagrangian:

$$V = \mu^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2 = -\lambda v^2\Phi^\dagger\Phi + \lambda(\Phi^\dagger\Phi)^2$$

Show that inserting

$$\Phi = \frac{1}{\sqrt{2}}e^{i\frac{\theta^a(x)}{v}\tau^a}\begin{pmatrix}0\\ v+h(x)\end{pmatrix}.$$

into V gives

$$V = \frac{\lambda}{4}\left(h^4 + 4h^3v - v^4 + 4hv(\theta_1^2+\theta_2^2+\theta_3^2) + (\theta_1^2+\theta_2^2+\theta_3^2)^2 + 4h^2v^2 + 2h^2(\theta_1^2+\theta_2^2+\theta_3^2)\right).$$

Solution

The product $\Phi^\dagger\Phi$ is simply

$$\Phi^\dagger\Phi = \frac{h^2+2hv+v^2+\theta_1^2+\theta_2^2+\theta_3^2}{2}$$

So,

$$V(\Phi^\dagger\Phi) = \lambda\left(\frac{1}{4}\left(h^2+2hv+v^2+\theta_1^2+\theta_2^2+\theta_3^2\right)^2 - \frac{v^2}{2}\left(h^2+2hv+v^2+\theta_1^2+\theta_2^2+\theta_3^2\right)\right)$$

It's easy to see that because of the minus sign, all θ_i^2 terms will cancel. Of course, h^2 term will also cancel, but there is more: we have a mixed $2hv$ term, which, after squaring, will become a term proportional to h^2 and thus, h field will gain mass. Finally,

$$V(\Phi^\dagger\Phi) = \frac{\lambda}{4}\left(h^4 + 4h^3v - v^4 + 4hv(\theta_1^2+\theta_2^2+\theta_3^2) + (\theta_1^2+\theta_2^2+\theta_3^2)^2 + 4h^2v^2 + 2h^2(\theta_1^2+\theta_2^2+\theta_3^2)\right)$$

III.2.8 [1 point(s)]

Examine the whole resulting Lagrangian. How does the number of degrees of freedom compare to that of the initial Lagrangian? What is the reason for this and how can it be resolved?

Solution

The whole Lagrangian can be written as

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}\left(\partial^\mu h\right)\left(\partial_\mu h\right) + \frac{1}{2}\left(\partial^\mu\theta_1\right)\left(\partial_\mu\theta_1\right) + \frac{1}{2}\left(\partial^\mu\theta_2\right)\left(\partial_\mu\theta_2\right) + \frac{1}{2}\left(\partial^\mu\theta_3\right)\left(\partial_\mu\theta_3\right)\\
&+ \frac{g}{2}W_\mu^1\left(h\partial^\mu\theta_1 + v\partial^\mu\theta_1 - \theta_1\partial^\mu h + \theta_3\partial^\mu\theta_2 - \theta_2\partial^\mu\theta_3\right)\\
&+ \frac{g}{2}W_\mu^2\left(h\partial^\mu\theta_2 + v\partial^\mu\theta_2 - \theta_2\partial^\mu h + \theta_1\partial^\mu\theta_3 - \theta_3\partial^\mu\theta_1\right)\\
&+ \frac{g}{2}W_\mu^3\left(h\partial^\mu\theta_3 + v\partial^\mu\theta_3 - \theta_3\partial^\mu h + \theta_2\partial^\mu\theta_1 - \theta_1\partial^\mu\theta_2\right)\\
&+ \frac{g^2}{8}\left(\left(W_\mu^1\right)^2 + \left(W_\mu^2\right)^2 + \left(W_\mu^3\right)^2\right)\left(v^2 + 2vh + h^2 + \theta_1^2 + \theta_2^2 + \theta_3^2\right)\\
&- \frac{\lambda}{4}h^4 - \lambda vh^3 - \lambda v^2h^2 + \frac{\lambda}{4}v^4\\
&- \lambda\left(hv + \frac{h^2}{2}\right)(\theta_1^2+\theta_2^2+\theta_3^2)\\
&- \frac{\lambda}{4}(\theta_1^2+\theta_2^2+\theta_3^2)^2\end{aligned}$$

At this point, Lagrangian seems to have more degrees of freedom than we started with: Gauge bosons W^a have now each gained an extra polarization (massless vector bosons only have 2 polarizations, but massive ones have 3). So, we have 3 more (unphysical) degrees of freedom. The resolution lies within the $SU(2)$ gauge freedom of Φ. In the next parts we will transform Φ in such a way that the fields $\theta_{1,2,3}$ vanish.

III.2.9 [1 point(s)]

Use gauge freedom to eliminate the θ fields completely from the Lagrangian.

Solution

We have our main field Φ written as

$$\Phi = \frac{1}{\sqrt{2}} e^{i\frac{\theta^a(x)}{v}\tau^a} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

We also know, that Φ has an $SU(2)$ gauge freedom, i.e. transformations

$$\Phi \to e^{i\frac{\alpha^a}{2}\tau^a} \Phi$$

leave the Lagrangian invariant. If we use this transformation:

$$\Phi \to e^{i\frac{\alpha^a}{2}\tau^a} \frac{1}{\sqrt{2}} e^{i\frac{\theta^a(x)}{v}\tau^a} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

and choose $\alpha^a = -2\theta^a/v$ (remember, gauge transformations are local, so α^a is a function of x, just like θ^a), that would transform Φ into

$$\Phi \to \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix}$$

Consequently, we will have no θ fields in the Lagrangian.

III.2.10 [1 point(s)]

What are the masses of the vector bosons after the elimination of the θ fields? How many degrees of freedom does the resulting Lagrangian have?

Solution

Eliminating the θ fields, we're left with

$$\begin{aligned}\mathcal{L} = &\frac{1}{2}(\partial^\mu h)(\partial_\mu h) - \frac{\lambda}{4}h^4 - \lambda v h^3 - \lambda v^2 h^2 + \frac{\lambda}{4}v^4 \\ &+ \frac{g^2}{8}\left((W_\mu^1)^2 + (W_\mu^2)^2 + (W_\mu^3)^2\right)(v^2 + 2vh + h^2)\end{aligned}$$

Here we can already see mass terms for gauge bosons W_μ^a:

$$\mathcal{L}_{M_W} = \frac{g^2v^2}{8}(W_\mu^1)^2 + \frac{g^2v^2}{8}(W_\mu^2)^2 + \frac{g^2v^2}{8}(W_\mu^3)^2$$

with $M_W = gv/2$.
As for the scalar field,

$$\mathcal{L}_{m_h} = -\lambda v^2 h^2$$

with $m_h = \sqrt{2\lambda}v$.
As for the degrees of freedom, since three scalar fields θ disappeared, the three additional degrees of freedom are no more present (degrees of freedom for the scalar fields have been ”transferred” to the longitudinal polarizations of W gauge bosons).

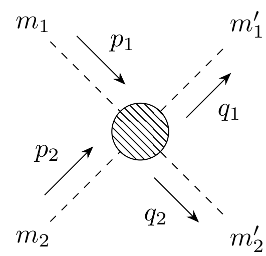
Appendix

Pauli matrices:

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

III.3 Electron-Positron to Pions (10 points)

Part 1: Kinematics



For a $2 \to 2$ process of spinless particles with initial momenta p_1, p_2 and final momenta q_1, q_2, the amplitude can depend only on the scalar products:

$$p_1^2,\ \ p_2^2,\ \ q_1^2,\ \ q_2^2,\ \ p_1 \cdot p_2,\ \ p_1 \cdot q_1,\ \ p_1 \cdot q_2,\ \ p_2 \cdot q_1,\ \ p_2 \cdot q_2,\ \ q_1 \cdot q_2.$$

III.3.1 [1 point(s)]

Give arguments why only 2 of these 10 scalars are independent. Where do the constraints come from?

Solution

The first 4 are constrained by the mass-shell conditions:

$$p_i^2 = m_i^2, \quad q_i^2 = m_i'^2, \qquad i = 1, 2.$$

Energy-momentum conservation gives 4 additional constraints:

$$p_1^\mu + p_2^\mu = q_1^\mu + q_2^\mu, \qquad \mu = 0, 1, 2, 3.$$

This fixes 8 out of 10 variables and therefore leaves 2 of them independent.
Alternatively, one can define three scalar quantities called the Mandelstam variables:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (q_1 + q_2)^2, \\ t &= (p_1 - q_1)^2 = (p_2 - q_2)^2, \\ u &= (p_1 - q_2)^2 = (p_2 - q_1)^2. \end{aligned}$$

As discussed above, only two of these variables can be linearly independent. In fact, it can be shown that s, t and u satisfy

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2.$$

III.3.2 [1 point(s)]

The n-particle phase space is defined as

$$d\Phi_n = \delta^{(4)}\left(\sum_i p_i - \sum_j q_j\right) \prod_{j=1}^{n} \frac{d^3 q_j}{(2\pi)^3 2E_{\vec{q}_j}},$$

The differential cross section for a $2 \to 2$ process is

$$d\sigma_{m_1 m_2 \to m_1' m_2'} = \frac{(2\pi)^4 \left|\langle q_1, q_2 | t | p_1, p_2 \rangle\right|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} d\Phi_2$$

Show that the full cross section is then

$$\sigma_{m_1 m_2 \to m'_1 m'_2} = \frac{1}{64\pi^2} \frac{\sqrt{\lambda(s, m_1'^2, m_2'^2)}}{\sqrt{\lambda(s, m_1^2, m_2^2)}} \frac{1}{s} \int \frac{|t_{fi}|^2}{\mathbb{S}} d\Omega_{\vec{q}}$$

where $\mathbb{S}$ is the symmetry factor and λ is the Källén function, defined as

$$\lambda\left(s, m_1^2, m_2^2\right) = \left(s - (m_1 + m_2)^2\right)\left(s - (m_1 - m_2)^2\right).$$

t_{fi} is the invariant amplitude of the process (the indices i and f stand for initial and final states, respectfully).

Solution

The differential cross section for a $2 \to 2$ process is

$$d\sigma_{m_1 m_2 \to m'_1 m'_2} = \frac{(2\pi)^4 |\langle q_1, q_2 | t | p_1, p_2 \rangle|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} d\Phi_2,$$

where initial and final states are denoted by $|p_1, p_2\rangle$ and $|q_1, q_2\rangle$, and $m_{1,2}$ are the masses of the particles in the initial state. Of course, they need not to be the same in the final state. We will denote final state masses with $m'_{1,2}$. Using the definition of $d\Phi_2$, we get for a two-body phase space

$$\begin{aligned}
\sigma_{m_1 m_2 \to m'_1 m'_2} &= \int \frac{(2\pi)^4 \delta^{(4)}(p_1 + p_2 - q_1 - q_2)}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{|t_{fi}|^2}{\mathbb{S}} \frac{d^3 q_1}{(2\pi)^3 2E_{\vec{q}_1}} \frac{d^3 q_2}{(2\pi)^3 2E_{\vec{q}_2}} \\
\xrightarrow{\text{CM Frame}} &= \frac{1}{16\pi^2} \int \frac{\delta(\sqrt{s} - E_{\vec{q}_1} - E_{\vec{q}_2}) \delta^{(3)}(\vec{q}_1 + \vec{q}_2)}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{|t_{fi}|^2}{\mathbb{S}} \frac{d^3 q_1}{E_{\vec{q}_1}} \frac{d^3 q_2}{E_{\vec{q}_2}} \\
&= \frac{1}{16\pi^2} \int \frac{\delta(\sqrt{s} - E_{m'_1, \vec{q}_1} - E_{m'_2, -\vec{q}_1})}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{|t_{fi}|^2}{\mathbb{S}} \frac{d^3 q_1}{E_{m'_1, \vec{q}_1} E_{m'_2, -\vec{q}_1}} \\
\xrightarrow{E_{m,\vec{q}_1} = E_{m,-\vec{q}_1} \equiv E_{m,\vec{q}}} &= \frac{1}{16\pi^2 \mathbb{S}} \int \frac{\delta(\sqrt{s} - E_{m'_1, \vec{q}} - E_{m'_2, \vec{q}})}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{|t_{fi}|^2 d^3 q}{E_{m'_1, \vec{q}} E_{m'_2, \vec{q}}} \\
&= \frac{1}{16\pi^2} \int \frac{\delta(\sqrt{s} - \sqrt{m_1^2 + \vec{q}^2} - \sqrt{m_2^2 + \vec{q}^2})}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \frac{|t_{fi}|^2}{\mathbb{S}} \frac{\vec{q}^2 d|\vec{q}| d\Omega_{\vec{q}}}{\sqrt{m_1^2 + \vec{q}^2}\sqrt{m_1^2 + \vec{q}^2}} \\
&= \frac{1}{16\pi^2 \mathbb{S}} \frac{\sqrt{\lambda(s, m_1'^2, m_2'^2)}}{2s} \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \int |t_{fi}|^2 d\Omega_{\vec{q}},
\end{aligned}$$

In the CM frame, where $\vec{p}_1 = -\vec{p}_2 \equiv \vec{p}$, the Källén function gives the solution of

$$\begin{aligned}
\sqrt{s} &= \sqrt{m_1 + \vec{p}^{\,2}} + \sqrt{m_2 + \vec{p}^{\,2}}, \\
\implies \quad \vec{p}^{\,2} &= \frac{\lambda(s, m_1^2, m_2^2)}{4s} = \frac{\left(s - (m_1 + m_2)^2\right)\left(s - (m_1 - m_2)^2\right)}{4s}.
\end{aligned}$$

The flux factor in the denominator is given by

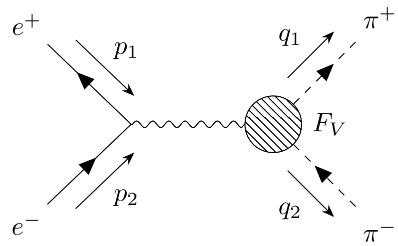
$$4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = 4\sqrt{s}|\vec{p}| = 2\sqrt{\lambda(s, m_1^2, m_2^2)}.$$

Therefore,

$$\sigma_{m_1 m_2 \to m'_1 m'_2} = \frac{1}{64\pi^2} \frac{\sqrt{\lambda(s, m_1'^2, m_2'^2)}}{\sqrt{\lambda(s, m_1^2, m_2^2)}} \frac{1}{s} \int \frac{|t_{fi}|^2}{\mathbb{S}} d\Omega_{\vec{q}}.$$

Part 2: $e^+e^- \to \pi^+\pi^-$

We consider the process $e^+e^- \to \pi^+\pi^-$, described by the following diagram:



We define the Mandelstam variables as follows:

$$\begin{aligned} s &= (p_1 + p_2)^2 = (q_1 + q_2)^2, \\ t &= (p_1 - q_1)^2 = (p_2 - q_2)^2, \\ u &= (p_1 - q_2)^2 = (p_2 - q_1)^2. \end{aligned}$$

Apart from that, let us define

$$\begin{aligned} k &= p_1 + p_2 = q_1 + q_2, \\ l &= p_1 - p_2, \quad l' = q_1 - q_2. \end{aligned}$$

III.3.3 [1 point(s)]

Give the expression for the leptonic current L^μ (left side of the diagram above) using the Feynman rules for QED.

Solution

The leptonic current is

$$L^\mu = \bar{v}^s(p_1)(-ie\gamma^\mu)u^r(p_2)$$

III.3.4 [1 point(s)]

The hadronic current H^μ (right side of the diagram above) can be written as

$$H^\mu = (q_1 + q_2)^\mu G_V(s) + (q_1 - q_2)^\mu F_V(s).$$

Argue, why $G_V(s)$ can be safely neglected here.

Solution

The leptonic current is transverse to $p_1 + p_2 = k = q_1 + q_2$. Consequently, the part of the hadronic current which is proportional to k^μ vanishes after contraction.

III.3.5 [1 point(s)]

Give the expression for the invariant amplitude $\mathcal{M}$ for the process.

Solution

$$\begin{aligned} i\mathcal{M} &= \bar{v}^s(p_1)(-ie\gamma^\mu)u^r(p_2)\frac{-ig_{\mu\nu}}{(p_1+p_2)^2}(ie)(q_1-q_2)^\nu F_V(s) \\ &= -i\frac{e^2}{s}\bar{v}^s(p_1)\gamma^\mu u^r(p_2)l'_\mu F_V(s). \end{aligned}$$

III.3.6 [1 point(s)]

Square the invariant amplitude, average out over all initial spins and sum over all final ones. Give the final result for the spin-averaged invariant matrix element squared $\overline{|\mathcal{M}|^2}$.

Solution

Taking the absolute value squared,

$$\begin{aligned} |\mathcal{M}|^2 &= \left(\frac{e^2}{s}\right)^2 (\bar{v}^s(p_1)\gamma^\mu u^r(p_2))^* \left(\bar{v}^s(p_1)\gamma^\nu u^r(p_2)\right) l'_\mu l'_\nu |F_V(s)|^2 \\ &= \left(\frac{e^2}{s}\right)^2 (\bar{u}^r(p_2)\gamma^\mu v^s(p_1)) \left(\bar{v}^s(p_1)\gamma^\nu u^r(p_2)\right) l'_\mu l'_\nu |F_V(s)|^2. \end{aligned}$$

Averaging over all initial spin states,

$$\begin{aligned} \overline{|\mathcal{M}|^2} &= \frac{1}{4}\sum_{s,r}\left(\frac{e^2}{s}\right)^2 (\bar{u}^r(p_2)\gamma^\mu v^s(p_1)) \left(\bar{v}^s(p_1)\gamma^\nu u^r(p_2)\right) l'_\mu l'_\nu |F_V(s)|^2 \\ &= \frac{1}{4}\sum_{s,r}\left(\frac{e^2}{s}\right)^2 \left(\bar{u}^r_\alpha(p_2)\gamma^\mu_{\alpha\beta}v^s_\beta(p_1)\right) \left(\bar{v}^s_\rho(p_1)\gamma^\nu_{\rho\sigma}u^r_\sigma(p_2)\right) l'_\mu l'_\nu |F_V(s)|^2 \\ &= \frac{1}{4}\sum_{s,r}\left(\frac{e^2}{s}\right)^2 \left(v^s_\beta(p_1)\bar{v}^s_\rho(p_1)u^r_\sigma(p_2)\bar{u}^r_\alpha(p_2)\gamma^\mu_{\alpha\beta}\gamma^\nu_{\rho\sigma}\right) l'_\mu l'_\nu |F_V(s)|^2 \\ &= \frac{1}{4}\left(\frac{e^2}{s}\right)^2 \left(\left(\not{p}_1 - m_e\right)_{\beta\rho}\left(\not{p}_2 + m_e\right)_{\sigma\alpha}\gamma^\mu_{\alpha\beta}\gamma^\nu_{\rho\sigma}\right) l'_\mu l'_\nu |F_V(s)|^2 \\ &= \frac{1}{4}\left(\frac{e^2}{s}\right)^2 \mathrm{tr}\left(\gamma^\mu\left(\not{p}_1 - m_e\right)\gamma^\nu\left(\not{p}_2 + m_e\right)\right) l'_\mu l'_\nu |F_V(s)|^2. \end{aligned}$$

III.3.7 [1 point(s)]

Calculate the following trace

$$\frac{1}{4}\mathrm{tr}\left(\gamma^\mu\left(\not{p}_1 - m_e\right)\gamma^\nu\left(\not{p}_2 + m_e\right)\right)$$

Hint

$\not{A} = \gamma^\mu A_\mu$.

Solution

$$
\begin{aligned}
\frac{1}{4}\mathrm{tr}\left(\gamma^{\mu}\left(\not{p}_1 - m_e\right)\gamma^{\nu}\left(\not{p}_2 + m_e\right)\right) &= \left(p_1^{\mu}p_2^{\nu} + p_1^{\nu}p_2^{\mu} - \frac{k^2}{2}g^{\mu\nu}\right) \\
&= \left(\frac{(k+l)^{\mu}(k-l)^{\nu}}{4} + \frac{(k+l)^{\nu}(k-l)^{\mu}}{4} - \frac{k^2}{2}g^{\mu\nu}\right) \\
&= \left(\frac{k^{\mu}k^{\nu}}{2} - \frac{l^{\mu}l^{\nu}}{2} - \frac{k^2}{2}g^{\mu\nu}\right) \\
&= -\frac{1}{2}\Big(\left(k^2 g^{\mu\nu} - k^{\mu}k^{\nu}\right) + \left(l^{\mu}l^{\nu}\right)\Big).
\end{aligned}
$$

III.3.8 [1 point(s)]

Express $\overline{|\mathcal{M}|^2}$ in terms of the Mandelstam variable s, the scattering angle θ_s, and the Källén function λ, where

$$
\begin{aligned}
\cos(\theta_s) &= \frac{t-u}{\kappa(s)}, \\
\kappa(s) &= \frac{\lambda^{1/2}(s, m_\pi^2, m_\pi^2)\lambda^{1/2}(s, m_e^2, m_e^2)}{s}, \\
\lambda\left(s, m_1^2, m_2^2\right) &= \left(s - (m_1+m_2)^2\right)\left(s - (m_1-m_2)^2\right)
\end{aligned}
$$

Solution

Using the resulting trace from above:

$$
\begin{aligned}
\overline{|\mathcal{M}|^2} &= -\frac{1}{2}\left(\frac{e^2}{s}\right)^2\Big(\left(k^2 g^{\mu\nu} - k^{\mu}k^{\nu}\right) + \left(l^{\mu}l^{\nu}\right)\Big)l'_{\mu}l'_{\nu}|F_V(s)|^2 \\
&= -\frac{1}{2}\left(\frac{e^2}{s}\right)^2\left(s\left(l'\right)^2 + \left(l\cdot l'\right)^2\right)|F_V(s)|^2 \\
&= -\frac{1}{2}\left(\frac{e^2}{s}\right)^2\left(s\left(4m_\pi^2 - s\right) + (t-u)^2\right)|F_V(s)|^2 \\
&= +\frac{1}{2}\left(\frac{e^2}{s}\right)^2\Big(\underbrace{s\left(s - 4m_\pi^2\right)}_{\lambda(s, m_\pi^2, m_\pi^2)} - \kappa^2(s)\cos^2(\theta_s)\Big)|F_V(s)|^2.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\overline{|\mathcal{M}|^2} &= \frac{1}{2}\left(\frac{e^2}{s}\right)^2\left(\lambda(s, m_\pi^2, m_\pi^2) - \lambda(s, m_\pi^2, m_\pi^2)\lambda(s, m_e^2, m_e^2)\frac{1}{s^2}\cos^2(\theta_s)\right)|F_V(s)|^2 \\
&= \frac{1}{2}\left(\frac{e^2}{s}\right)^2\lambda(s, m_\pi^2, m_\pi^2)\left(1 - \frac{\lambda(s, m_e^2, m_e^2)}{s^2}\cos^2(\theta_s)\right)|F_V(s)|^2.
\end{aligned}
$$

III.3.9 [1 point(s)]

Integrate $\overline{|\mathcal{M}|^2}$ over the solid angle to obtain $\int \overline{|\mathcal{M}|^2} d\Omega$.

Solution

$$\begin{aligned}\int \overline{|\mathcal{M}|^2} d\Omega &= (2\pi) \int_{-1}^{+1} \left(\frac{e^2}{s}\right)^2 \lambda(s, m_\pi^2, m_\pi^2) \frac{1}{2} \left(1 - \frac{\lambda(s, m_e^2, m_e^2)}{s^2} \cos^2(\theta_s)\right) |F_V(s)|^2 d\cos(\theta_s) \\ &= \frac{2\pi e^4}{s^2} \lambda(s, m_\pi^2, m_\pi^2) \underbrace{\left(1 - \frac{1}{3} \frac{\lambda(s, m_e^2, m_e^2)}{s^2}\right)}_{=2/3 \text{ for } m_e \ll s} |F_V(s)|^2.\end{aligned}$$

III.3.10 [1 point(s)]

Calculate the total cross section $\sigma_{e^+e^- \to \pi^+\pi^-}$.

Hint

You may use the limit $m_e^2 \ll s$.

Solution

Using the expression for a 2-by-2 cross-section, one obtains

$$\begin{aligned}\sigma_{e^+e^- \to \pi^+\pi^-} &= \frac{1}{64\pi^2} \frac{\lambda^{1/2}(s, m_\pi^2, m_\pi^2)}{\lambda^{1/2}(s, m_e^2, m_e^2)} \frac{1}{s} \int \overline{|\mathcal{M}|^2} d\Omega \\ &= \frac{2\pi e^4}{64\pi^2 s^3} \frac{\lambda^{3/2}(s, m_\pi^2, m_\pi^2)}{\lambda^{1/2}(s, m_e^2, m_e^2)} \left(1 - \frac{1}{3} \frac{\lambda(s, m_e^2, m_e^2)}{s^2}\right) |F_V(s)|^2 \\ \xrightarrow[\alpha = e^2/4\pi]{m_e^2 \ll s} &= \frac{32\pi^3 \alpha^2}{64\pi^2 s^3} \frac{\lambda^{3/2}(s, m_\pi^2, m_\pi^2)}{s} \left(1 - \frac{1}{3}\right) |F_V(s)|^2 \\ &= \frac{\pi \alpha^2}{3} \frac{\lambda^{3/2}(s, m_\pi^2, m_\pi^2)}{s^4} |F_V(s)|^2.\end{aligned}$$

IV Other

IV.1 Breaking Classical Mechanics (5 points)

Construction of Quantum mechanics from Classical mechanics usually begins with a process known as Quantization. This is usually done by constructing a map which takes observables to operators, that is:

$$\{f, g\} \to -\frac{i}{\hbar}[\hat{f}, \hat{g}]$$

Where $\{-, -\}$ is the Poisson bracket and $[-, -]$ is the commutator. One of the common properties of these brackets is that they form a Lie algebra, that is, they satisfy the following properties:

1. The bracket $[-, -]$ is billinear.
2. For any f, g we have $[f, g] = -[g, f]$
3. The bracket satisfies Jacobi identity, that is, for any f, g, h we have:

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0$$

IV.1.1 [5 point(s)]

Suppose that we have a "broken" classical mechanics in 3 dimensions, where $\{-, -\}$ doesn't satisfy the Jacobi identity. Prove that the resulting quantum mechanics would violate Heisenberg's uncertainty principle.

Solution

$$\{f, g\} = \omega(X_f, X_g)$$

it is easy to show that if $\{-, -\}$ doesn't satisfy Jacobi identity, then ω is not closed (which is possible since ω is not a top form), and thus is not symplectic. Hence, we can choose a Hamiltonian vector field: $X \in \text{Ham}(M, \omega)$ s.t. we have:

$$\mathcal{L}_X\omega = \iota_X d\omega + d\iota_X\omega = \iota_X d\omega \neq 0$$

Then, for volume form we obtain $\text{vol}_M = \omega^n$ where n is s.t. dim $M = 2n = 6$, hence, we see that $\mathcal{L}_X\text{vol}_M \neq 0$. Taking a volume element $U \subset M$ s.t. it is of minimal volume: Vol(U) $= (\hbar/2)^3$, and letting $g^t = \exp(tX)$, we obtain:

$$\frac{d}{dt}\text{Vol}(g^t U) = \frac{d}{dt}\int_U g^{t*}\mathcal{L}_X\omega^3 \neq 0$$

that there exists an appropriate choice for $X \in \text{Ham}(M, \omega)$ s.t.:

$$\frac{d}{dt}\text{Vol}(g^t U) < 0$$

is trivial. Hence, there exists time $t > 0$ s.t.

$$\text{Vol}(g^t U) < \hbar^3/8$$

which is less than the minimum volume, thus violating the Heisenberg's uncertainty principle.

IV.2 The Green's Function (5 points)

Consider the Green's function in three dimensions:

$$G(\vec{x}) = -\frac{e^{ik|\vec{x}|}}{4\pi|\vec{x}|}.$$

IV.2.1 [2 point(s)]

Show $(\Delta + k^2)G(\vec{x}) = 0$ for $\vec{x} \neq 0$, where Δ is the Laplace operator.

Solution

The radial part of the Laplacian in spherical coordinates (here $|\vec{x}| \equiv r$) is written as

$$\Delta_r = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}.$$

Consequently,

$$\begin{aligned}
(\Delta + k^2)G(\vec{x}) &= -\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + k^2\right)\frac{e^{ikr}}{4\pi r} \\
&= -\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\frac{ike^{ikr}}{4\pi r} - \frac{e^{ikr}}{4\pi r^2}\right) + k^2\frac{e^{ikr}}{4\pi r}\right) \\
&= -\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{ikre^{ikr}}{4\pi} - \frac{e^{ikr}}{4\pi}\right) + k^2\frac{e^{ikr}}{4\pi r}\right) \\
&= -\left(\frac{1}{r^2}\left(\frac{ike^{ikr}}{4\pi} - \frac{k^2re^{ikr}}{4\pi}\right) - \frac{1}{r^2}\frac{ike^{ikr}}{4\pi} + k^2\frac{e^{ikr}}{4\pi r}\right) \\
&= 0
\end{aligned}$$

IV.2.2 [3 point(s)]

Show that $G(\vec{x})$ satisfies the inhomogeneous differential equation

$$(\Delta + k^2)G(\vec{x}) = \delta^{(3)}(\vec{x}).$$

Hint

Consider the integral

$$\int_{|\vec{x}|\leq 1} d^3x(\Delta + k^2)G(\vec{x}).$$

Solution

Let us consider the integral.

$$\begin{aligned}
\int_{|\vec{x}|\le 1} d^3x(\Delta+k^2)G(\vec{x}) &= -\int_{|\vec{x}|\le 1} d^3x \left(\vec{\nabla}\left(\vec{\nabla}\frac{e^{ikr}}{4\pi r}\right)+k^2\frac{e^{ikr}}{4\pi r}\right) \\
\xrightarrow{\text{Stokes' theorem}} &= -\int_{|\vec{x}|\le 1} d\vec{S}\left(\vec{\nabla}\frac{e^{ikr}}{4\pi r}\right) - \int_0^1 r^2 dr\, k^2 \frac{e^{ikr}}{r} \\
&= -\int_{|\vec{x}|\le 1} d\vec{S}\frac{d}{dr}\left(\frac{e^{ikr}}{4\pi r}\right)\underbrace{(\vec{\nabla}r)}_{\vec{x}/r} - k^2\int_0^1 r dr\, e^{ikr} \\
&= -\int_{|\vec{x}|\le 1} \underbrace{d\vec{S}\frac{\vec{x}}{r}}_{d\Omega}\frac{d}{dr}\left(\frac{e^{ikr}}{4\pi r}\right) - k^2\int_0^1 r dr\, e^{ikr} \\
\xrightarrow[\text{boundary}]{r=1\text{ at the}} &= -\int_{|\vec{x}|\le 1} d\Omega\left(\frac{ike^{ik}}{4\pi}-\frac{e^{ik}}{4\pi}\right) - k^2\left(\frac{re^{ikr}}{ik}\Big|_0^1 - \int_0^1 dr\frac{e^{ikr}}{ik}\right) \\
\xrightarrow{\int d\Omega=4\pi} &= -\left(ike^{ik}-e^{ik}-ike^{ik}+e^{ik}-1\right) \\
&= 1 \\
&= \int_{|\vec{x}|\le 1} d^3x\, \delta^{(3)}(\vec{x})
\end{aligned}$$

IV.3 Point group D_6 (5 points)

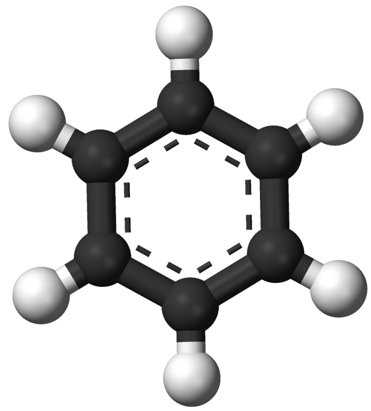


Figure 4: A molecule with D_6 symmetry. Credits: Wikipedia.

Consider the dihedral group $D_6 = \langle b, c \mid b^2 = c^6 = (bc)^2 = e \rangle$, which is the symmetry group for an unoriented hexagon.

IV.3.1 [1 point(s)]

D_6 has 6 conjugacy classes. One element per class is given below:

$$\begin{aligned} \mathcal{C}\ell_1 &= \{e, \dots\}, \\ \mathcal{C}\ell_2 &= \{c, \dots\}, \\ \mathcal{C}\ell_3 &= \left\{c^2, \dots\right\}, \\ \mathcal{C}\ell_4 &= \left\{c^3, \dots\right\}, \\ \mathcal{C}\ell_5 &= \{b, \dots\}, \\ \mathcal{C}\ell_6 &= \{bc, \dots\}. \end{aligned}$$

Complete the classes by adding corresponding elements within. Show why a specific element should belong to a specific class.

Hint

Not all $\{\dots\}$ are meant to be filled.

Solution

$\mathcal{C}\ell_1$ and $\mathcal{C}\ell_4$ contain single elements each, because e and c^3 compose the center of the group.
Using $c^{-n}b = bc^n$, we can show that b is conjugated with bc^2 and c^2b:

$$\begin{aligned} c^{-1}(b)c &= bc^2, \\ c^{-2}(b)c^2 &= bc^4. \end{aligned}$$

Similarly,

$$\begin{aligned} c^{-1}(bc)c &= bc^3, \\ c^{-2}(bc)c^2 &= bc^5. \end{aligned}$$

As for the other classes, c^2 is conjugated with c^4 by b (and the same holds for c and c^5:

$$b(c^2)b = bbc^-2 = c^-2 = c^4$$
$$b(c)b = bbc^-1 = c^-1 = c^5$$

With this, the complete list of classes is

$$\mathcal{C}\ell_1 = \{e\}\,,\ \mathcal{C}\ell_2 = \left\{c, c^5\right\}\,,\ \mathcal{C}\ell_3 = \left\{c^2, c^4\right\}\,,$$
$$\mathcal{C}\ell_4 = \left\{c^3\right\}\,,\ \mathcal{C}\ell_5 = \left\{b, bc^2, bc^4\right\}\,,\ \mathcal{C}\ell_6 = \left\{bc, bc^3, bc^5\right\}\,.$$

IV.3.2 [2 point(s)]

Let $\nu = 1, \ldots, 6$ enumerate irreducible representations of D_6 and d_ν denote the dimension of the representation. We consider a 6-dimensional representation $D^{(7)}$. The characters for the irreducible representations are given in Table 1. Fill the table by calculating the characters for $D^{(7)}$.

$D^{(\nu)}$	d_ν	$\mathcal{C}\ell_1$	$\mathcal{C}\ell_2$	$\mathcal{C}\ell_3$	$\mathcal{C}\ell_4$	$\mathcal{C}\ell_5$	$\mathcal{C}\ell_6$
$D^{(1)}$	1	1	1	1	1	1	1
$D^{(2)}$	1	1	-1	1	-1	-1	1
$D^{(3)}$	1	1	-1	1	-1	1	-1
$D^{(4)}$	1	1	1	1	1	-1	-1
$D^{(5)}$	2	2	1	-1	-2	0	0
$D^{(6)}$	2	2	-1	-1	2	0	0
$D^{(7)}$	6	...	...	...	...	...	...

Table 1: Character table for D_6.

Hint

Don't get confused by the notation: D_6 stands for the dihedral group. $D^{(\nu)}$ stand for specific representations.

Solution

Since every element within a class has the same character, it's sufficient to calculate the trace of a single element per class. One way to do this is to actually look at the matrix forms of the elements. For the identity and the generators we have:

$$D^{(7)}(e) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad D^{(7)}(b) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad D^{(7)}(c) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

So,

$$\chi^7(\mathcal{C}\ell_1) = tr(D^{(7)}(e)) = 6,$$
$$\chi^7(\mathcal{C}\ell_5) = tr(D^{(7)}(b)) = 2,$$
$$\chi^7(\mathcal{C}\ell_2) = tr(D^{(7)}(c)) = 0.$$

Using the matrix forms above, c^2, bc and bc^2 can be constructed to show

$$\chi^7(\mathcal{C}\ell_3) = tr(D^{(7)}(c^2)) = 0,$$
$$\chi^7(\mathcal{C}\ell_4) = tr(D^{(7)}(bc)) = 0,$$
$$\chi^7(\mathcal{C}\ell_6) = tr(D^{(7)}(bc^2)) = 0.$$

IV.3.3 [1 point(s)]

Using the characters derived in the previous section, decompose $D^{(7)}$ into irreducible representations.

Solution

The multiplicity q_ν of an irreducible representation $D^{(\nu)}$ within $D^{(7)}$ is given by

$$q_\nu = \frac{1}{|D_6|} \sum_{i=1}^{6} N_i \left(\chi^\nu(\mathcal{C}\ell_i)\right)^* \chi^7(\mathcal{C}\ell_i),$$

where N_i is the number of elements in class i and $\chi^\nu(\mathcal{C}\ell_i)$ is the character of the class i within representation $D^{(\nu)}$. Using the table, we obtain

$$q_1 = 1,\ \ q_2 = 0,\ \ q_3 = 1,\ \ q_4 = 0,\ \ q_5 = 1,\ \ q_6 = 1.$$

Finally, we can write

$$D^{(7)} = D^{(1)} \oplus D^{(3)} \oplus D^{(5)} \oplus D^{(6)}.$$

IV.3.4 [1 point(s)]

Consider a molecule with D_6 symmetry, which transforms under $D^{(7)}$ (an example is given on Figure 4). What can you deduce about the energy levels (and their degeneracies) of this molecule, judging from the decomposition of $D^{(7)}$?

Solution

Since $D^{(7)}$ decomposes into 4 irreducible representations, we expect four different energy levels, with two of them doubly degenerate (since two irreps have dimension 2). The trivial representation is expected to correspond to the ground state (lowest energy level).

IV.4 Dirac Equation: Angular Momentum and Parity (5 points)

The Dirac equation for spin-1/2 particles is written as

$$i\partial_t \psi = \hat{H}\psi,$$

where

$$\hat{H} := \alpha^i \left(\hat{p}_i - qA_i\right) + \beta m + \mathbb{I}q\Phi.$$

Here $\alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}$.

We assume that the electric field is time-independent and rotationally invariant:

$$V(\mathbf{x}) := q\Phi = V(r).$$

We take the vector potential to be vanishing: $A_i = 0$. This simplifies the Hamiltonian to

$$\hat{H} = \alpha^i \hat{p}_i + \beta m + \mathbb{I}V$$

We combine the angular momentum and the spin operators

$$\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$$
$$S_i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

to obtain the total angular momentum operator

$$\hat{J}_i = \mathbb{I}\hat{L}_i + S_i.$$

IV.4.1 [1 point(s)]

Show that the commutation relations for $\hat{J}$ are

$$\left[\hat{J}_i, \hat{J}_j\right] = i\epsilon_{ijk}\hat{J}_k, \qquad \left[\hat{J}_i, \hat{J}^2\right] = 0$$

Solution

First of all, let's take a look at the commutation relations for $\hat{L}$ and S separately,

$$\left[\hat{L}_i, \hat{L}_j\right] = i\epsilon_{ijk}\hat{L}_k$$
$$\left[S_i, S_j\right] = i\epsilon_{ijk}S_k$$

Both these relations result from the definitions of $\hat{L}, S$. Also, it's clear that these two operators commute, since they act on different spaces. So,

$$\left[\hat{J}_i, \hat{J}_j\right] = \mathbb{I}\left[\hat{L}_i, \hat{L}_j\right] + \left[S_i, S_j\right]$$
$$= i\epsilon_{ijk}\mathbb{I}\hat{L}_k + i\epsilon_{ijk}S_k = i\epsilon_{ijk}\hat{J}_k$$

As for the casimir operator,

$$\begin{aligned}\left[\hat{J}_i, \hat{J}_j\hat{J}_j\right] &= \hat{J}_j\left[\hat{J}_i, \hat{J}_j\right] + \left[\hat{J}_i, \hat{J}_j\right]\hat{J}_j \\ &= i\epsilon_{ijk}\left(\hat{J}_j\hat{J}_k + \hat{J}_k\hat{J}_j\right) = 0\end{aligned}$$

This is zero, because ϵ_{ijk} is totally antisymmetric and multiplying it on a symmetric object gives zero.

IV.4.2 [2 point(s)]

Show that $\hat{J}_i$ and $\hat{J}^2$ commute with the given Hamiltonian:

$$\left[\hat{H}, \hat{J}_i\right] = 0, \qquad \left[\hat{H}, \hat{J}^2\right] = 0.$$

Solution

By construction of our Hamiltonian,

$$\begin{aligned}\left[\hat{H}, \hat{J}_k\right] &= \left[\alpha^i\hat{p}_i + \beta m + \mathbb{I}V, \hat{J}_k\right] \\ &= \left[\alpha^i\hat{p}_i, \hat{J}_k\right] + \left[\beta m, \hat{J}_k\right] + \left[\mathbb{I}V, \hat{J}_k\right] \\ &= \left[\alpha^i\hat{p}_i, \hat{J}_k\right] + \left[\beta m, \hat{J}_k\right]\end{aligned}$$

The third part disappeared because unity commutes with everything.
We calculate the rest separately,

$$\begin{aligned}\left[\alpha^i\hat{p}_i, \hat{J}_k\right] &= \left[\alpha^i\hat{p}_i, \mathbb{I}\hat{L}_k + S_k\right] \\ &= \alpha^i\left[\hat{p}_i, \hat{L}_k\right] + \left[\alpha^i, S_k\right]\hat{p}_i\end{aligned}$$

$$\begin{aligned}\left[\hat{p}_i, \hat{L}_k\right] &= \left[\hat{p}_i, \epsilon_{kmn}\hat{x}_m\hat{p}_n\right] \\ &= \epsilon_{kmn}\left[\hat{p}_i, \hat{x}_m\right]\hat{p}_n \\ &= \epsilon_{kmn}(-i\delta_{im})\hat{p}_n \\ &= -i\epsilon_{kin}\hat{p}_n\end{aligned}$$

$$\begin{aligned}\left[\alpha^i, S_k\right] &= \frac{1}{2}\left[\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0\end{pmatrix}, \begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}\right] \\ &= \frac{1}{2}\begin{pmatrix} 0 & \sigma^i\sigma^k - \sigma^k\sigma^i \\ \sigma^i\sigma^k - \sigma^k\sigma^i & 0\end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & i\epsilon_{ikn}\sigma^n \\ i\epsilon_{ikn}\sigma^n & 0\end{pmatrix} \\ &= i\epsilon_{ikn}\alpha^n\end{aligned}$$

So, finally,

$$\begin{aligned}\left[\alpha^i\hat{p}_i, \hat{J}_k\right] &= \alpha^i\left[\hat{p}_i, \hat{L}_k\right] + \left[\alpha^i, S_k\right]\hat{p}_i \\ &= -i\alpha^i\epsilon_{kin}\hat{p}_n + i\epsilon_{ikn}\alpha^n\hat{p}_i = 0\end{aligned}$$

As for the second term,

$$\begin{aligned}\left[\beta m, \hat{J}_k\right] &= \left[\beta m, \mathbb{I}\hat{L}_k + S_k\right] \\ &= \left[\beta m, S_k\right] \\ &= \frac{m}{2}\left[\begin{pmatrix}\mathbb{I} & 0 \\ 0 & -\mathbb{I}\end{pmatrix}, \begin{pmatrix}\sigma^k & 0 \\ 0 & \sigma^k\end{pmatrix}\right] = 0\end{aligned}$$

So, in all, $\hat{J}_k$ commutes with the Hamiltonian:

$$\left[\hat{H}, \hat{J}_k\right] = 0$$

As for the square operator,

$$\left[\hat{H}, \hat{J}_k\hat{J}_k\right] = \hat{J}_k\left[\hat{H}, \hat{J}_k\right] + \left[\hat{H}, \hat{J}_k\right]\hat{J}_k = 0$$

IV.4.3 [2 point(s)]

Parity operator for spinors is defined as $\hat{P}_s := \beta\hat{P}$ and acts as

$$\hat{P}_s\psi(t, \mathbf{x}) = \beta\psi(t, -\mathbf{x})$$

Show that this operator commutes with $\hat{H}$, $\hat{J}_i$, and $\hat{J}^2$.

Solution

Hamiltonian and the angular momentum operators are built by the following set of operators:

$$\left\{\mathbb{I}\hat{x}_i, \mathbb{I}\hat{p}_i, \mathbb{I}\hat{L}_i, S_i\right\}$$

So, we check the commutations with these operators first:

$$\begin{aligned}
\left[\mathbb{I}\hat{x}_i, \hat{P}_s\right]\psi(t, \mathbf{x}) &= \hat{x}_i\hat{P}_s\psi(t, \mathbf{x}) - \hat{P}_s\hat{x}_i\psi(t, \mathbf{x}) \\
&= \hat{x}_i\beta\hat{P}\psi(t, \mathbf{x}) - \beta\hat{P}\hat{x}_i\psi(t, \mathbf{x}) \\
&= \hat{x}_i\beta\psi(t, -\mathbf{x}) - \beta(-\hat{x}_i\psi(t, -\mathbf{x})) \\
&= \hat{x}_i\beta\psi(t, -\mathbf{x}) + \beta\hat{x}_i\psi(t, -\mathbf{x}) \\
&= 2\hat{x}_i\beta\psi(t, -\mathbf{x}) \\
&= 2\hat{x}_i\hat{P}_s\psi(t, \mathbf{x}) \\
\left[\mathbb{I}\hat{x}_i, \hat{P}_s\right] &= 2\hat{x}_i\hat{P}_s \\
\left\{\mathbb{I}\hat{x}_i, \hat{P}_s\right\} &= 0
\end{aligned}$$

Similarly, $\hat{p}_i$, being component of a non-axial vector, will also change sign under $\hat{P}$ and the results will be the same:

$$\begin{aligned}
\left[\mathbb{I}\hat{p}_i, \hat{P}_s\right] &= 2\hat{p}_i\hat{P}_s \\
\left\{\mathbb{I}\hat{p}_i, \hat{P}_s\right\} &= 0
\end{aligned}$$

Conversely, $\hat{L}_i$ is a component of an axial vector. It consists of both $\hat{x}_i$ and $\hat{p}_i$. So, it will not change direction.

$$\begin{aligned}
\left[\mathbb{I}\hat{L}_i, \hat{P}_s\right]\psi(t, \mathbf{x}) &= \hat{L}_i\hat{P}_s\psi(t, \mathbf{x}) - \hat{P}_s\hat{L}_i\psi(t, \mathbf{x}) \\
&= \hat{L}_i\beta\hat{P}\psi(t, \mathbf{x}) - \beta\hat{P}\hat{L}_i\psi(t, \mathbf{x}) \\
&= \hat{L}_i\beta\psi(t, -\mathbf{x}) - \beta\hat{L}_i\psi(t, -\mathbf{x}) \\
&= 0 \\
\left[\mathbb{I}\hat{L}_i, \hat{P}_s\right] &= 0 \\
\left\{\mathbb{I}\hat{L}_i, \hat{P}_s\right\} &= 2\hat{L}_i\hat{P}_s
\end{aligned}$$

The spin operator S_i does not have any spatial components and it commutes with the β matrix. So, consequentially,

$$\begin{aligned}
\left[S_i, \hat{P}_s\right]\psi(t, \mathbf{x}) &= S_i\hat{P}_s\psi(t, \mathbf{x}) - \hat{P}_sS_i\psi(t, \mathbf{x}) \\
&= S_i\beta\hat{P}\psi(t, \mathbf{x}) - \beta\hat{P}S_i\psi(t, \mathbf{x}) \\
&= S_i\beta\psi(t, -\mathbf{x}) - \beta S_i\psi(t, -\mathbf{x}) \\
&= 0 \\
\left[S_i, \hat{P}_s\right] &= 0 \\
\left\{S_i, \hat{P}_s\right\} &= 2S_i\hat{P}_s
\end{aligned}$$

These two alone suffice to deduce that $\left[\hat{J}_i, \hat{P}_s\right] = 0$, and consequently, $\left[\hat{J}^2, \hat{P}_s\right] = 0$.
As for the Hamiltonian,

$$\begin{aligned}
\left[\hat{H}, \hat{P}_s\right] &= \left[\alpha^i \hat{p}_i + \beta m + \mathbb{I}V, \hat{P}_s\right] \\
&= \left[\alpha^i \hat{p}_i, \hat{P}_s\right] \\
&= \alpha^i \left[\hat{p}_i, \hat{P}_s\right] + \left[\alpha^i, \hat{P}_s\right] \hat{p}_i \\
&= 2\alpha^i \hat{p}_i \hat{P}_s + \left[\alpha^i, \hat{P}_s\right] \hat{p}_i \\
&= 2\alpha^i \hat{p}_i \hat{P}_s + \left(\alpha^i \beta \hat{P} - \beta \alpha^i \hat{P}\right) \hat{p}_i \\
&= 2\alpha^i \hat{p}_i \hat{P}_s + \left(\alpha^i \beta \hat{P} + \alpha^i \beta \hat{P}\right) \hat{p}_i \\
&= 2\alpha^i \hat{p}_i \hat{P}_s + 2\alpha^i \beta \hat{P} \hat{p}_i \\
&= 2\alpha^i \left\{\hat{p}_i, \hat{P}_s\right\} = 0
\end{aligned}$$

IV.5 Transverse Magnetic Susceptibility of an Isotropic Ferromagnet (5 points)

In an isotropic ferromagnet, the ground state with all the spins polarized in the same direction is infinitely degenerate. The ground state manifold represents a sphere whose points correspond to possible directions of the spontaneous magnetization

$$\vec{M} = N^{-1} \sum_{i=1}^{N} \langle \vec{S}_i \rangle \,.$$

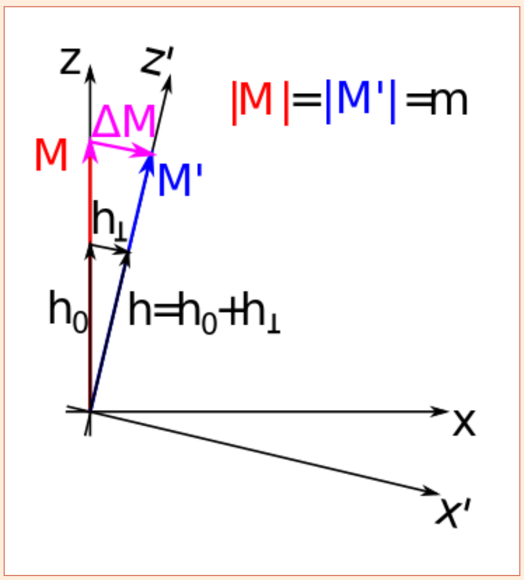
In an external magnetic field $\vec{h}_0$ the magnetization $\vec{M}$ will be aligned along $\vec{h}_0$. A small transverse magnetic field $\vec{h}_\perp$ (with $\vec{h}_\perp \cdot \vec{h}_0 = 0$) will slightly change the direction of $\vec{M}$.

IV.5.1 [3 point(s)]

Calculate the transverse magnetic susceptibility of the ferromagnet

$$\chi_\perp(h_0) = \lim_{h_\perp \to 0} \frac{\partial M(h_0; h_\perp)}{\partial h_\perp}.$$

Solution



The magnetization in the ground state manifold only differs by the direction and the amplitude is fixed, thus $|M| = |M'| =$ const. Since $|\vec{h}_\perp| \ll |\vec{h}_0|$, then $\Delta\vec{M}$ can also be regarded as perpendicular to $\vec{M}$. This way, due to the similarity of triangles given in the figure above, we have

$$\frac{|\Delta M|}{|h_\perp|} = \frac{|M'|}{|h|} = \frac{|M|}{\sqrt{h_0^2 + h_\perp^2}}$$

$$\vec{M}'(\vec{h}_0, \vec{h}_\perp) = \vec{M} + \Delta\vec{M}(\vec{h}_0, \vec{h}_\perp)$$

In it's ground state:

$$\vec{M} = \max_{\text{all spin configurations}} \{|M|\}\hat{z}$$

.

$$\chi|_{h_\perp=0} = \left.\frac{\partial \Delta M(\vec{h}_0, \vec{h}_\perp)}{\partial h_\perp}\right|_{h_\perp=0} = |M| \frac{\partial}{\partial h_\perp}\left.\left(\frac{h_\perp}{\sqrt{h_0^2 + h_\perp^2}}\right)\right|_{h_\perp=0} = \frac{|M|}{|h_0|}$$

IV.5.2 [2 point(s)]

What is the property of $\chi_\perp$ in the limit $h_0 = 0$? Explain the result.

Solution

The transverse magnetic susceptibility diverges in the limit $h_0 = 0$, since in this limit the rotational symmetry of the problem is completely recovered and the system becomes extremely sensitive to arbitrarily small magnetic field.

IV.6 Gaussian Integrals (5 points)

IV.6.1 [2 point(s)]

Let A be a real, symmetric, positive definite matrix. Show the following identity for multi-dimensional integrals over real variables x_i:

$$\int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x_k A_{kl} x_l + J_k x_k\right) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left(\frac{1}{2} J_k A^{-1}_{jl} J_l\right).$$

Solution

Introduce notation

$$\begin{aligned} x &= (x_1, \ldots, x_n) \\ y &= (y_1, \ldots, y_n) \\ J &= (J_1, \ldots, J_n) \end{aligned}$$

In this notation, our identity looks like this:

$$\int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T A x + J^T x\right) = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left(\frac{1}{2} J^T A^{-1} J\right)$$

Since A is symmetric, we can diagonalize it using orthogonal matrices. And because of the fact that it's positive definite, this orthogonal matrix will have determinant 1. Since orthogonal matrices satisfy $O^T O = \mathbb{I}_n$, we can write

$$A = O^T A_D O$$

First, let's set $J = 0$, so we're calculating

$$\begin{aligned} \int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T A x\right) &= \int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T O^T A_D O x\right) \\ \xrightarrow{y \equiv Ox} &= \int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}y^T A_D y\right) \\ \xrightarrow{\det(O)=1} &= \int \prod_{i=1}^{n} dy_i \exp\left(-\frac{1}{2}y^T A_D y\right) \\ &= \int \prod_{i=1}^{n} dy_i \exp\left(-\frac{1}{2}\sum_{i=1}^{n} \lambda_i y_i^2\right) \\ &= \int dy_1 \exp\left(-\frac{1}{2}\lambda_1 y_1^2\right) \cdots \int dy_n \exp\left(-\frac{1}{2}\lambda_n y_n^2\right) \\ &= \sqrt{\frac{2\pi}{\lambda_1}} \cdots \sqrt{\frac{2\pi}{\lambda_n}} = \frac{(2\pi)^{n/2}}{\sqrt{\lambda_1 \cdots \lambda_2}} = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \end{aligned}$$

Now, let's put J back on:

$$\begin{aligned}
&\int \prod_{i=1}^{n} dx_i \exp\left(-\frac{1}{2}x^T A x + J^T x\right) = \\
\xrightarrow[dx_i = dy_i]{x \equiv y+b} &= \int \prod_{i=1}^{n} dy_i \exp\left(-\frac{1}{2}(y^T + b^T)A(y+b) + J^T(y+b)\right) \\
&= \int \prod_{i=1}^{n} dy_i \exp\left(-\frac{1}{2}y^T A y - \frac{1}{2}b^T A y - \frac{1}{2}y^T A b - \frac{1}{2}b^T A b + J^T y + J^T b\right) \\
\xrightarrow{A^T = A} &= \int \prod_{i=1}^{n} dy_i \exp\left(-\frac{1}{2}y^T A y - b^T A y - \frac{1}{2}b^T A b + J^T y + J^T b\right) \\
&= \int \prod_{i=1}^{n} dy_i \exp\left(-\frac{1}{2}y^T A y - (b^T A - J^T)y - \frac{1}{2}b^T A b + J^T b\right) \\
\xrightarrow[\text{by setting } b = A^{-1}J]{b^T A \overset{!}{=} J^T} &= \int \prod_{i=1}^{n} dy_i \exp\left(-\frac{1}{2}y^T A y - \frac{1}{2}b^T A b + b^T A b\right) \\
&= \int \prod_{i=1}^{n} dy_i \exp\left(-\frac{1}{2}y^T A y\right) \exp\left(+\frac{1}{2}b^T A b\right) \\
&= \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left(+\frac{1}{2}b^T A b\right) \\
\xrightarrow{b = A^{-1}J} &= \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \exp\left(+\frac{1}{2}J^T A^{-1} J\right)
\end{aligned}$$

IV.6.2 [3 point(s)]

Show that for complex variables z_i, the previous result can be generalized as follows:

$$\int \prod_{i=1}^{n} dz_i^* dz_i \exp\left(-z_k^* H_{kl} z_l + J_k^* z_k + J_k z_k^*\right) = \frac{(2\pi i)^n}{\det H} \exp\left(J_k^* H_{kl}^{-1} J_l\right)$$

where H is now hermitian, positive definite matrix.

Solution

As before, let's start with $J = 0$ case.
Since H is hermitian, it can be diagonalized using unitary matrices, which satisfy $U^\dagger U = \mathbb{I}_n$. Since it's also positive definite, there is a special unitary matrix (i.e. with determinant 1) that satisfies this:

$$H = U^\dagger H_D U$$

So,

$$\begin{aligned}
\int \prod_{i=1}^{n} dz_i^* dz_i \exp\left(-z^\dagger H z\right) &= \int \prod_{i=1}^{n} dz_i^* dz_i \exp\left(-z^\dagger U^\dagger H_D U z\right) \\
\xrightarrow{w \equiv Uz} &= \int \prod_{i=1}^{n} dz_i^* dz_i \exp\left(-w^\dagger H_D w\right) \\
\xrightarrow{\det U = 1} &= \int \prod_{i=1}^{n} dw_i^* dw_i \exp\left(-w^\dagger H_D w\right) \\
&= \int \prod_{i=1}^{n} dw_i^* dw_i \exp\left(-\lambda_i w_i^* w_i\right) \\
&= \prod_{i=1}^{n} \int dw_i^* dw_i \exp\left(-\lambda_i w_i^* w_i\right)
\end{aligned}$$

Since the integrals factorize, we only need to calculate one of them. Also, note that the sloppy notation of $dw_i^* dw_i$ is, in reality,

$$\begin{aligned}
dw_i^* \wedge dw_i &= d(x_i - iy_i) \wedge d(x_i + iy_i) \\
&= dx_i \wedge dx_i + dx_i \wedge idy_i - idy_i \wedge dx_i - dy_i \wedge dy_i \\
\xrightarrow[\text{property}]{\text{antisymmetric}} &= 2i\, dx_i \wedge dy_i \\
&\equiv 2i\, dx_i dy_i \xleftarrow[\text{sloppy notation}]{\text{reintroducing the}}
\end{aligned}$$

Also, noting that $w_i^* w_i = x_i^2 + y_i^2$, we have

$$\begin{aligned}
\int dw_i^* dw_i \exp\left(-\lambda_i w_i^* w_i\right) &= 2i \int dx_i \exp\left(-\lambda_i x_i^2\right) \int dy_i \exp\left(-\lambda_i y_i^2\right) \\
&= 2i \sqrt{\frac{\pi}{\lambda_i}} \sqrt{\frac{\pi}{\lambda_i}} = \frac{2\pi i}{\lambda_i}
\end{aligned}$$

Taking the product, we have

$$\begin{aligned}
\int \prod_{i=1}^{n} dz_i^* dz_i \exp\left(-z^\dagger H z\right) &= \prod_{i=1}^{n} \int dw_i^* dw_i \exp\left(-\lambda_i w_i^* w_i\right) \\
&= \prod_{i=1}^{n} \frac{2\pi i}{\lambda_i} = \frac{(2\pi i)^n}{\lambda_1 \cdots \lambda_n} = \frac{(2\pi i)^n}{\det H}
\end{aligned}$$

To go on, we introduce $z = w + b$ and proceed as we did above:

$$\begin{aligned}
z_k^* H_{kl} z_l - J_k^* z_k - J_k z_k^* &= z^\dagger H z - J^\dagger z - z^\dagger J \\
&= (w^\dagger + b^\dagger) H (w + b) - J^\dagger (w + b) - (w^\dagger + b^\dagger) J \\
&= w^\dagger H w + b^\dagger H w + w^\dagger H b + b^\dagger H b - J^\dagger w - J^\dagger b - w^\dagger J - b^\dagger J \\
&= w^\dagger H w + (b^\dagger H - J^\dagger) w + w^\dagger (H b - J) + b^\dagger H b - J^\dagger b - b^\dagger J \\
\xrightarrow[\text{setting}]{b = H^{-1} J} &= w^\dagger H w + b^\dagger H b - J^\dagger b - b^\dagger J \\
&= w^\dagger H w + J^\dagger H^{-1} J - J^\dagger H^{-1} J - J^\dagger H^{-1} J \\
&= w^\dagger H w - J^\dagger H^{-1} J
\end{aligned}$$

Putting this into our integral, we arrive at the final result:

$$\begin{aligned}
\int \prod_{i=1}^{n} dz_i^* dz_i \exp\left(-z_k^* H_{kl} z_l + J_k^* z_k + J_k z_k^*\right) &= \int \prod_{i=1}^{n} dz_i^* dz_i \exp\left(-z^\dagger H z + J^\dagger z + z^\dagger J\right) \\
&= \int \prod_{i=1}^{n} dw_i^* dw_i \exp\left(-w^\dagger H w\right) \exp\left(J^\dagger H^{-1} J\right) \\
&= \frac{(2\pi i)^n}{\det H} \exp\left(J^\dagger H^{-1} J\right)
\end{aligned}$$